# Towards *p*-adic periods in Chabauty-Kim theory

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Work in progress.

Joint with D. Corwin.

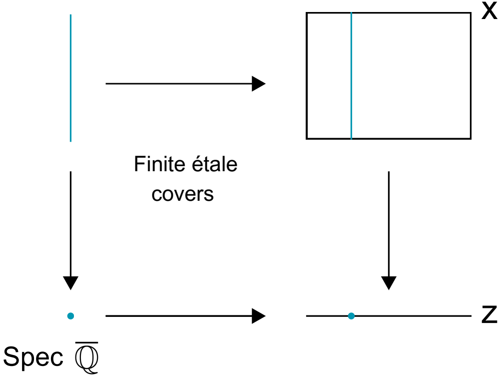
- We'll focus on the case of the complement of the zero section inside an elliptic scheme without CM over an open subscheme of $\operatorname{Spec} \mathbb{Z}$:

$$X \subset E \to Z \subset \operatorname{Spec} \mathbb{Z}.$$

- Grothendieck's theory of the étale fundamental group suggests one way to think about the sections of

$$X \to Z$$

  ("integral points").
- Kim's approach to the study of integral points suggests a different way.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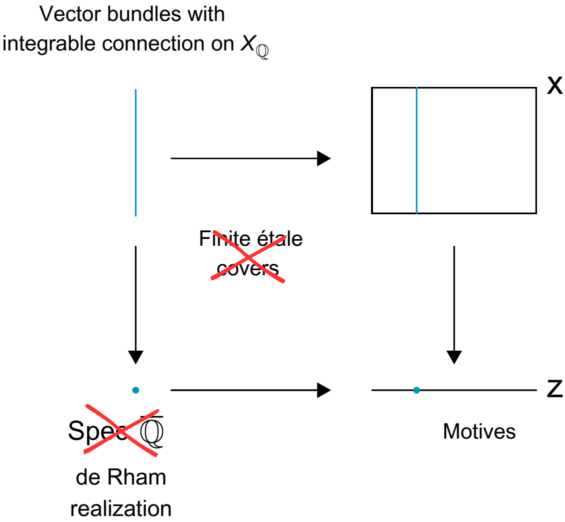
Vector bundles with
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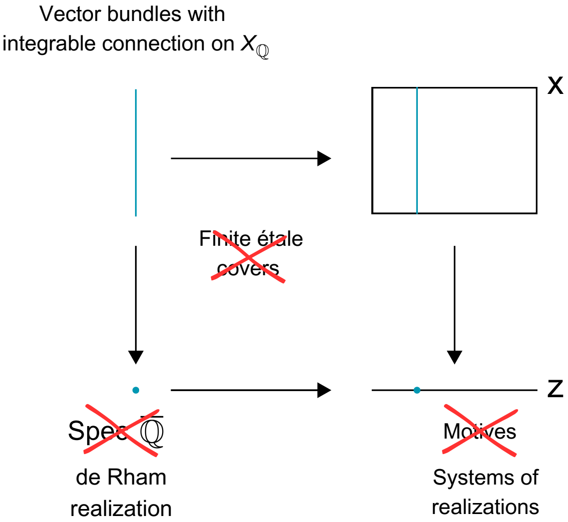
Vector bundles with
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Systems of
realizations

# Systems of realizations

- There are many realizations of $H_1(E)$: a mixed Hodge structure, Galois representations, a filtered $\phi$ module; these are related by various comparison isomorphisms.
- Better:

$$H_1(E) \in \mathrm{MMSys}(Z)$$

belongs to the Tannakian category of *mixed motivic systems of realizations over $Z$ with $\mathbb{Q}$ coefficients*.

# Mixed elliptic systems and the unipotent fundamental group

- Let
$$\mathrm{M_E Sys}(Z) \subset \mathrm{MMSys}(Z)$$
be the Tannakian subcategory generated by $H_1(E)$, closed under extensions.
- We fix a point $b$ for use as base-point.
- The *unipotent fundamental group* $\tilde{U}(X, b)$ is a prounipotent group object in $\mathrm{M_E Sys}(Z)$. We'll actually work with a unipotent quotient $U(X, b)$.

# Properties

- The Betti realization $\tilde{U}(X, b)_B$ is the prounipotent completion of the topological fundamental group.
- The de Rham realization $\tilde{U}(X, b)_{\mathrm{dR}}$ is the Tannakian fundamental group of the category of unipotent vector bundles with integrable connection on $X_{\mathbb{Q}}$.
- Fix a prime $p \in Z$. The *filtered $\phi$ realization* $U(X, b)_{F\phi}$ is the de Rham realization over $\mathbb{Q}_p$ equipped with some naturally occurring extra structures: a "Hodge" filtration and a "Frobenius" automorphism.
- A point $x \in X(Z)$ gives rise to a torsor object $U(X, b, x)$.

# Special geometric paths

- The $p$-adic de Rham realization of the path torsor $U(X, b, x)_{F\phi}$ is defined more generally for $x \in X(\mathbb{Z}_p)$. The subspace

$$F^0 U(X, b, x)_{F\phi} \subset U(X, b, x)_{F\phi}$$

  forms a torsor under $F^0 U(X, b)_{F\phi}$
- (these are the "geometric Hodge paths").
- There's a unique Frob. fixed path $p^{cr} \in U(X, b, x)_{F\phi}$
- (the "geometric crystalline path").
- The *unipotent p-adic Albanese map*

$$\alpha : X(\mathbb{Z}_p) \to F^0 \backslash U(X, b)_{F\phi}$$

  is given by

$$\alpha(x) = (p^H)^{-1} \circ p^{cr}$$

  for any geometric Hodge path $p^H \in F^0 U(X, x, b)_{F\phi}$.

# Relationship to iterated integrals

- An element $\omega \in T^\bullet H^1(E)_{\mathrm{dR}}$ of the tensor algebra may be used to define a function
$$f_\omega : U(X,b)_{\mathrm{dR}} \to \mathbb{A}^1_{\mathbb{Q}};$$
- if $f_\omega$ factors through $F^0\backslash U(X,b)$, then the composition
$$X(\mathbb{Z}_p) \xrightarrow{\alpha} F^0\backslash U(X,b)_{F\phi} \xrightarrow{f_\omega} \mathbb{A}^1_{\mathbb{Q}_p}$$
is given by the $p$-adic iterated integral
$$x \mapsto \int_b^x \omega.$$

# Structure of arithmetic Tannakian Galois group

- The Tannakian Galois group $G(\mathrm{M_E Sys}, \mathrm{dR}) = G(\mathrm{dR}) = G(Z)$ sits in a noncanonically split short exact sequence

$$1 \to U(Z) \to G(Z) \to \mathbb{G} \to 1$$

with $\mathbb{G}$ reductive and $U(Z)$ prounipotent,
- the "unipotent fundamental group of $Z$".
- The sequence becomes canonically split if we replace $\mathrm{dR}$ by $\mathrm{dR} \circ \mathrm{gr}^W$.
- Under our assumption that $E$ does *not* have CM, $\mathbb{G} = GL_2$.
- We let

$$F^0 U(\mathrm{dR}) \subset U(\mathrm{dR})$$

be the subgroup of Tannakian loops which preserve Hodge filtrations.

# Axiomatic setting for Theorems A, B, C

- The category $\mathrm{M_E Sys}(Z)$ is an example of a *weight filtered Tannakian category* $T$. (Objects are equipped with filtrations, morphisms are strict, pure objects are semisimple.)
- De Rham realization is in a natural way a *Hodge filtered fiber functor* $\omega^H : T \to \mathsf{FilVect}(\mathbb{Q})$. In this axiomatic setting, let's denote the associated fiber functor (de Rham realization) by $\omega$.
- Crystalline realization is in a natural way a *Frobenius equivariant fiber functor*

$$\omega^{cr} : T \to \phi\,\mathsf{Vect}(\mathbb{Q}_p).$$

(The operator $\phi$ is required to act with eigenvalues of weight $n$ on $\mathrm{gr}_n^W M$.)

# Theorem (A)

- *The space of $\otimes$-compatible natural transformations*

$$T \underset{\omega^H}{\overset{\omega^H \circ \mathrm{gr}^W}{\underset{\Downarrow\, \mathfrak{p}^H}{\rightrightarrows}}} \mathrm{FilVect}(\mathbb{Q})$$

  *which induce the identity on associated graded objects, forms a trivial torsor under $F^0 U(\omega)$.*
- *These are our "arithmetic Hodge paths".*

## Theorem (B)

- *There exists a unique ⊗-compatible natural transformation*

$$T \;\underset{\omega^{cr}}{\overset{\omega^{cr}\circ \mathrm{gr}^{W}}{\rightrightarrows}}\; \phi\,\mathrm{Vect}(\mathbb{Q}_p) \qquad \Downarrow \mathfrak{p}^{cr}$$

*which induces the identity on associated graded objects.*

- *This is our "arithmetic crystalline path".*

- We fix an arithmetic Hodge path $\mathfrak{p}^H$.
- :(
- We define the *unipotent p-adic period loop* by

$$\mathfrak{u} := \mathfrak{p}^{cr} \circ (\mathfrak{p}^H)^{-1}.$$

- Existence and uniqueness of geometric crystalline paths applies in our axiomatic setting to torsor objects under effective prounipotent group objects in $T$.

# Theorem C

- Let $\pi$ be an effective unipotent group object in $T$ (concentrated in negative weights)
- and let $P$ be a torsor object with Frobenius fixed point $p^{cr}$.
- Then
$$\mathfrak{u}^{-1} p^{cr} \in F^0 P_\omega.$$

- Said differently, our unipotent $p$-adic period loops interchange geometric crystalline paths and geometric Hodge paths.
- Thus, the choice of arithmetic Hodge path gives us consistent choices of geometric Hodge paths.
- Wherever we encounter a geometric crystalline path $p^{cr}$, we set

$$p^H := \mathfrak{u}^{-1} p^{cr}.$$

- Our main application takes place in an intermediate level of generality.
- For instance, $X$ may be a suitable model of a hyperbolic curve with motivic Mumford-Tate group $\mathbb{G}$ (fundamental group of the category of pure motivic systems generated by $H_1$).

- We obtain the following

## Corollary

*We have a commutative diagram like so:*

$$\begin{array}{ccc} X(Z) & \longrightarrow & X(\mathbb{Z}_p) \\ \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle \alpha} \\ Z^1(U(Z,\mathrm{dR}), U(X)_{\mathrm{dR}})^{\mathbb{G}} & \xrightarrow[ev_{\mathfrak{u}}]{} & F^0\backslash U(X)_{F\phi}. \end{array}$$

- Here

$$\kappa_x(\gamma) = (p^H)^{-1} \circ \gamma(p^H),$$

- and $ev_{\mathfrak{u}}$ denotes evaluation at $\mathfrak{u}$.

- We also have a factorization of the evaluation map

$$\begin{array}{ccc} Z^1(U(Z,\mathrm{dR}), U(X)_{\mathrm{dR}})^{\mathbb{G}} & \xrightarrow{ev_{\mathfrak{u}}} & F^0\backslash U(X)_{F\phi} \\ \downarrow & \nearrow_{ev_{\mathfrak{u}_{p\text{-ét}}}} & \\ Z^1\big(U(\mathrm{GalRep}), U(X)_{p\text{-ét}}\big)^{\mathbb{G}} & & \end{array}$$

associated to realization into a suitable category of $p$-adic Galois representations.
- $Z^1\big(U(\mathrm{GalRep}), U(X)_{p\text{-ét}}\big)^{\mathbb{G}}$ is the same as Kim's Selmer variety, so in particular representable by a finite type affine $\mathbb{Q}_p$-scheme.
- $Z^1(U(Z,\mathrm{dR}), U(X)_{\mathrm{dR}})^{\mathbb{G}}$ is conjecturally representable by a finite type affine $\mathbb{Q}$-variety. (But if we want to really compute the image of $ev_{\mathfrak{u}}$ then representability might not play a central role anyway.)

- We have

$$\text{Kim functions} = \alpha^\sharp \ker ev^\sharp_{\mathfrak{u}_{p\text{-ét}}}$$

$$\cup$$

$$\text{motivic Kim functions} = \alpha^\sharp \ker ev^\sharp_{\mathfrak{u}}.$$

- These are locally analytic functions on $X(\mathbb{Z}_p)$ which vanish on $X(Z)$.
- Our construction should make (motivic) Kim functions computationally more accessible from purely abelian input.
- Both variants have advantages and disadvantages.

We'll give two examples of concrete applications of our work.

# Example

- Joint work with D. Corwin and M. Lüdtke. Let $Z = \operatorname{Spec} \mathbb{Z}[1/2]$, $X = \mathbb{P}^1 \backslash \{0, 1, \infty\}$.
- Then $X(\mathbb{Z}_p)$ admits the motivic Kim function

$$\begin{aligned} \log(2)\zeta(3)\operatorname{Li}_{2,1,1}(z) - \left(\zeta(3)\log(2) - \frac{8}{7}\operatorname{Li}_4(2)\right)\operatorname{Li}_1(z)\operatorname{Li}_{2,1}(z) \\ - \tfrac{1}{24}\left(\log(2)\zeta(3) - 4\frac{8}{7}\operatorname{Li}_4(2)\right)\log(z)\operatorname{Li}_1(z)^3 \\ + \zeta(3)\left(\zeta(3)\log(2) - \frac{8}{7}\operatorname{Li}_4(2)\right)\operatorname{Li}_1(z). \end{aligned}$$

- This function occurs in weight 8, and hence beyond the so-called *quadratic quotient*.
- In a different direction, it goes beyond the so-called *polylogarithmic quotient*.

- The word in 1-forms

$$\omega = \frac{dt}{1-t} \cdot \frac{dt}{t} \cdot \frac{dt}{t}$$

  may be used to define a function

$$f_\omega : U(X, 0, 1) \to \mathbb{A}^1_{\mathbb{Q}}$$

  (tangential base points).
- Precomposing with the orbit map of a geometric Hodge path (unique in this case), we obtain a function

$$\zeta^{\mathfrak{f}}(3) : U(\mathsf{MTM}, \mathrm{dR}) \to \mathbb{A}^1$$

  (a "formal period")
- such that

$$\zeta^{\mathfrak{f}}(3)(\mathfrak{u}) = \zeta(3).$$

- This means that $\zeta(3)$ is a *p-adic period of a mixed Tate motive*.
- The Kim function above is a polynomial in Coleman functions that are in a certain sense rationally defined with period coefficients.

# Example

- Work of D. Corwin, contributions by O. Patashnik and M. Lüdtke.
- Suppose in the example

$$X \subset E \to Z \subset \operatorname{Spec} \mathbb{Z}$$

  we have 1 inverted prime and $E$ has $p$-Selmer rank 1.
- Let $\alpha, \beta$ be a basis of $H^1_{\mathrm{dR}}(E_{\mathbb{Q}})$ with $\alpha$ holomorphic and $\beta$ of type II.
- Then there exists a nonzero Kim function of the form

$$c_1 \int_0^x (\alpha\beta\beta - 2\beta) + c_2 \int_0^x \alpha\beta\alpha + c_3 \int_0^x (2\alpha\alpha\beta + \alpha\beta\alpha)$$

$$+ c_4 \int_0^x \alpha\alpha\alpha + c_5 \int_0^x \alpha.$$

Proof.

- We have
$U(Z, \mathrm{dR})^{\mathrm{ab}} = \prod_{i,j} \mathsf{Ext}^1\left(\mathbb{Q}(0), \mathsf{Sym}^i H_1(E)(j)\right)^\vee \otimes \mathsf{Sym}^i H_1^{\mathrm{dR}}(E)(j)$.
- There's a motivic quotient $U(X)$ of the unipotent fundamental group such that

$$\mathsf{Lie}\, U(X) = H_1(E) \oplus \mathbb{Q}(1) \oplus H_1(E)(1) =: W.$$

- In the category $\mathrm{M_E GRep}(Z)$ of mixed elliptic $p$-adic Galois representations which are unramified over $Z \backslash \{p\}$ and crystalline at $p$, we have

$$\begin{aligned} \dim \mathsf{Ext}^1(\mathbb{Q}_p(0), H_1(E)) &= \dim \mathsf{Ext}^1(\mathbb{Q}_p(0), \mathbb{Q}_p(1)) \\ &= \dim \mathsf{Ext}^1(\mathbb{Q}_p(0), H_1(E)(1)) = 1. \end{aligned}$$

- These calculations allow us to replace the unipotent Tannakian Galois group by the free prounipotent group with $\mathsf{GL}_{2,\mathbb{Q}_p}$-action $U(W_{p\text{-ét}})$.

$$
\begin{array}{ccc}
X(Z) & \longrightarrow & X(\mathbb{Z}_p) \\
\Big\downarrow{\scriptstyle \kappa} & & \Big\downarrow{\scriptstyle \alpha} \\
Z^1(U(Z), U(X)_{p-\text{ét}})^{\mathbb{G}} & \xrightarrow[ev_{\mathfrak{u}}]{} & F^0\backslash U(X)_{F\phi} \\
\Big\downarrow & & \Big\downarrow \\
U(Z) \times Z^1(U(Z), U(X)_{p-\text{ét}})^{\mathbb{G}} & \xrightarrow[ev]{} & U(Z) \times F^0\backslash U(X)_{F\phi}
\end{array}
$$

Using the above, we eventually reduce to the following purely geometric problem.

- Let $V$ be the standard representation of $GL_2$ over a field $k$.
- The representation

$$W = V \oplus \det \oplus V \otimes \det,$$

  carries a natural structure of nilpotent Lie algebra.
- Let $U^g$ be the associated unipotent group.
- Let

$$F^0 U^g \subset U^g$$

  be a subgroup given by certain explicit equations (work of J. Beacom).
- Let $U^a$ be free prounipotent on $W$.

- Finally, consider the universal evaluation map

$$\mathfrak{ev} : Z^1(U^a, U^g)^{\mathrm{GL}_2} \times U^a \to U^g \times U^a$$

$$\mathfrak{ev}(c, u) = (c(u), u).$$

- **Problem:** Find a finite free $\mathcal{O}(U^a)$-submodule $I$ of

$$\mathcal{O}(F^0 \backslash U^g) \otimes \mathcal{O}(U^a)$$

  such that $\mathfrak{ev}^\sharp I$ is contained in a finite free $\mathcal{O}(U^a)$-submodule of

$$\mathcal{O}(Z^1(U^a, U^g)^{\mathrm{GL}_2}) \otimes \mathcal{O}(U^a)$$

  of strictly lower rank.
- This sort of problem is easily solved in minimal examples,
- e.g. in the setting of the present example.

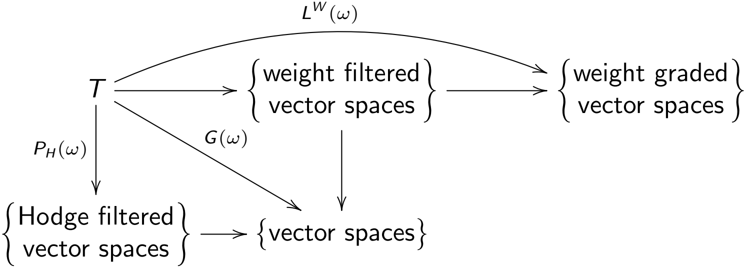
Proof of Theorem B (existence and uniqueness of arithmetic crystalline paths).

- Let $G(\omega)$ act on $U(\omega)$ by conjugation.
- We find that $U(\omega)$ is concentrated in negative weights.
- By Besser's argument, the Lang map

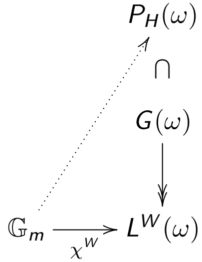
$$\begin{aligned} U(\omega) &\to U(\omega) \\ g &\mapsto g^{-1}\phi(g) \end{aligned}$$

  is iso.
- This implies that every $G(\omega)$-equivariant $U(\omega)$-torsor possesses a unique $\phi$-fixed point.
- We realize $\mathfrak{p}^{\mathrm{cr}}$ as the unique $\phi$-fixed point in a certain $G(\omega)$-equivariant $U(\omega)$-torsor.

Proof of Theorem A (existence of arithmetic Hodge paths).
We consider the following groups of ⊗-automorphisms of symmetric monoidal functors:



Via Tannaka duality, the problem translates into the problem of lifting an associated central cocharacter:




The kernel of the vertical map is unipotent. Work of P. Ziegler shows that in the finite type case, the groups on the right all have the same rank.

# Unipotent $p$-adic periods

- Our unipotent $p$-adic period loops give us a well defined point

$$\mathfrak{u} : \operatorname{Spec} \mathbb{Q}_p \to U(\mathrm{dR})/F^0.$$

- Equivalently, a homomorphism defined on a certain subalgebra

$$\mathbb{Q}_p \xleftarrow{\text{per}} \mathcal{O}\big(U(\mathrm{dR})/F^0\big) \subset \mathcal{O}\big(U(\mathrm{dR})\big).$$

- We call elements in its image *unipotent p-adic periods*.

# Example

- If $\alpha$ is a global nonvanishing 1-form on $E_{\mathbb{Q}}$ normalized with respect to a nonzero tangent vector $v$ at the origin,
- and $x \in E(\mathbb{Q})$,
- then there's a function

$$f_{x,\alpha} : U(\mathrm{dR})/F^0 \to \mathbb{A}^1_{\mathbb{Q}}$$

- such that

$$f_{x,\alpha}(\mathfrak{u}) = \int_v^x \alpha.$$

# Conjecture

- A natural extension of the $p$-adic period conjecture for mixed Tate motives says that the ring homomorphism

$$\mathsf{per} : \mathcal{O}\big(U(\mathrm{dR})/F^0\big) \to \mathbb{Q}_p$$

  is injective.
- Compare recent work by Ancona–Frăţilă.