A motivic Weil height machine for curves

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Work in progress.

Joint with L. Alexander Betts.

Table of Contents

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- Grothendieck envisioned a two step approach to understanding the rational points of a hyperbolic curve over a number field:

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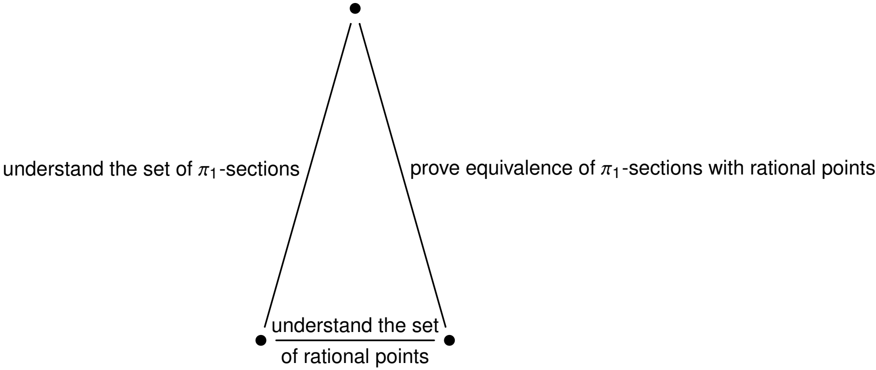
- Of course, we have Mordell $=$ Faltings but we still have neither of the above, including even the finiteness of the set of π_1-sections.

- In fact, one could argue that today, the picture looks rather more like this:



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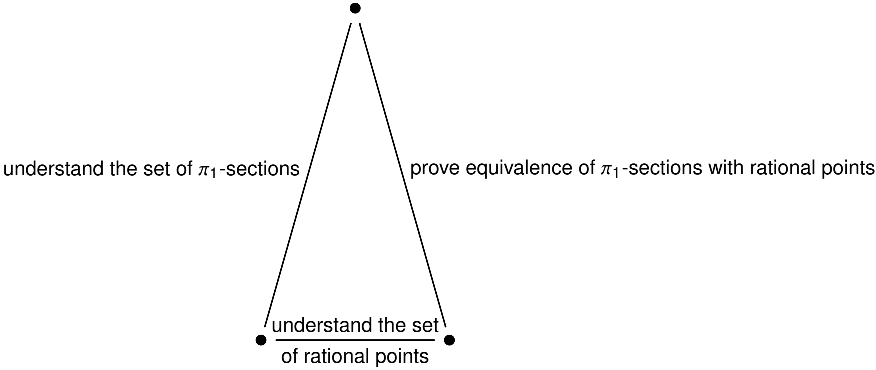

- While we do have an aspect of the bottom arrow by Faltings, there's still much to be desired (effective versions, uniform bounds, higher dimensions...).

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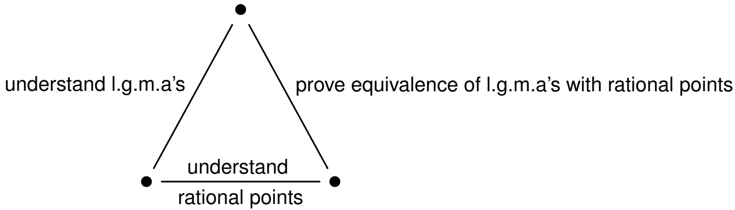

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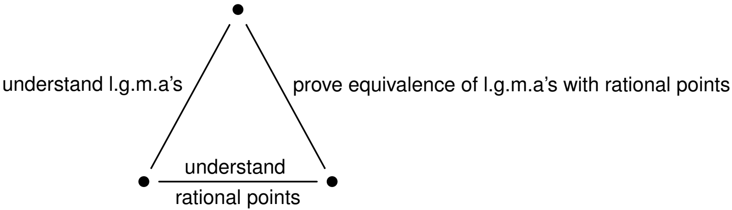

To give a sense of what we can show before explaining what these are, we have, for instance, the following.

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- Suppose there exists an abelian variety A such that

$$\operatorname{rank} \operatorname{Hom}(X, A) > 2 \operatorname{rank} A(K).$$

- Then for any morphism $X \to J$ to an abelian variety, the image of

$$\operatorname{Aug}^{\mathrm{lg}}(X) \to \operatorname{Aug}(J)$$

is finite.

Table of Contents

- There's a functor

$$C : \mathsf{Sm}_Z^{\mathrm{op}} \to \mathsf{CAlg}\,\mathsf{DM}(Z, \mathbb{Q})$$

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- There are essentially well defined continuous composition maps.
- The *homotopy category ho$\mathcal{D}$* is given by

$$\mathrm{Hom}_{hoD}(E, F) := \pi_0 \,\mathrm{Hom}(E, F).$$

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of topological spaces, the *homotopy pullback*

$$D = \lim(*)$$

consists of a natural topology on the set

$$D = \left\{(a, b, \gamma) \,\middle|\, a \in A, b \in B, \gamma \text{ a path } f(a) \xrightarrow{\sim} g(b) \text{ in } C\right\}.$$

- Limits and colimits in ∞-categories are determined by homotopy limits and colimits of topological spaces. For instance, colimits are determined by homotopy equivalences

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- A *stable* ∞-category $\mathcal{D}$ (like $\mathrm{DM}(Z, \mathbb{Q})$) possesses an object 0 which is both initial and terminal.
- A square diagram is a pullback diagram iff it's a pushout diagram.

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- Pairs of Cartesian squares of the form

$$\begin{array}{ccccc} E & \longrightarrow & F & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & G & \longrightarrow & \Sigma E \end{array}$$

give $ho\mathcal{D}$ the structure of a triangulated category.

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  - and about $\mathbb{G}_m$-torsors:

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This allows us to extend the theory of Néron-Tate heights on abelian varieties to motivic augmentations.

Table of Contents

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  - If in addition $M_0 \in \mathrm{Pic}^0(A)$, then there's a constant $C > 0$ such that

$$|\hat{h}_{M_0}(\alpha)| \leq C \cdot \sqrt{\hat{h}_M(\alpha)}.$$

- Depending on the choice of a vector $\tilde{0}$ in the fiber of M above 0, there are local “Néron-Tate height functions”

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- such that if

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$$\tilde{\alpha} \to \alpha$$

then

$$\hat{h}_M(\alpha) = \sum_v \lambda_{M,v}(\tilde{\alpha}).$$

If v is any place of K, then there's a commuting square

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Theorem (Motivic Weil height machine)

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such that if $f : X \to A$ is a morphism to an abelian variety and $M \in \mathrm{Pic}\, A$, then

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We may equally work with augmentations defined over an algebraic closure.

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are $\mathbb{Q}$-*linearly independent*. This means that under the conditions of the theorem, rational points have bounded height. Our version of Nothcott's theorem says that such a set has finite image in $\mathrm{Aug}(J)$ for any abelian variety J.

Table of Contents

The de Rham dga of an abelian variety $A_{\mathbb{C}}$ over $\mathbb{C}$ is equivalent to the cdga with zero differential

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A similar statement holds motivically:

- Let $\mathcal{D} = \mathrm{DM}(Z, \mathbb{Q})$ ($Z = \operatorname{Spec} K$) and let

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Thus,

$$\mathcal{A}\mathrm{ug}(A) = \mathrm{Hom}_{\mathcal{D}}\big(M^1(A), \mathbb{Q}(0)\big) = \mathrm{Hom}_{\mathcal{D}}(\mathbb{Q}(0), M_1(A)) = A(K) \otimes \mathbb{Q}.$$

Table of Contents

If $Y_{\mathbb{C}}$ is smooth over $\mathbb{C}$ and $L_{\mathbb{C}} \to Y_{\mathbb{C}}$ is a line bundle with first Chern class $c_{\mathrm{dR}} \in H^2_{\mathrm{dR}}(Y_{\mathbb{C}})$ and associated $\mathbb{G}_m$-torsor L^*,

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is the cofiber of c_{dR} then the induced square

$$\begin{array}{ccc} \mathrm{Sym}\, C_{\mathrm{dR}}(Y_{\mathbb{C}}) & \longrightarrow & \mathrm{Sym}\, E \\ \downarrow & & \downarrow \\ C_{\mathrm{dR}}(Y_{\mathbb{C}}) & \longrightarrow & C_{\mathrm{dR}}(L^*_{\mathbb{C}}) \end{array}$$

If $Y_{\mathbb{C}}$ is smooth over $\mathbb{C}$ and $L_{\mathbb{C}} \to Y_{\mathbb{C}}$ is a line bundle with first Chern class $c_{\mathrm{dR}} \in H^2_{\mathrm{dR}}(Y_{\mathbb{C}})$ and associated $\mathbb{G}_m$-torsor L^*, then

$$\Omega^\bullet_{C^\infty}(L^*) \simeq \Omega^\bullet_{C^\infty}(Y)[t]/(dt - c_{\mathrm{dR}}).$$

More precisely, if

$$\mathbb{C}[-2] \xrightarrow{c_{\mathrm{dR}}} C^*_{\mathrm{dR}}(Y_{\mathbb{C}}) \to E$$

is the cofiber of c_{dR} then the induced square

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is a homotopy pushout square.

Now let Y be smooth over K and L a line bundle over Y.

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Theorem (The motivic cochain algebra of a $\mathbb{G}_m$-bundle)

The square

$$\begin{array}{ccc} \mathrm{Sym}\, C(Y) & \longrightarrow & \mathrm{Sym}\, E \\ \downarrow & & \downarrow \\ C(Y) & \longrightarrow & C(L^*) \end{array}$$

is (homotopy) coCartesian.

- Hence, the augmentation space sits in a homotopy pullback square

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathcal{D}}\big(C(Y),\mathbb{Q}(0)\big) & \leftarrow & \mathrm{Hom}_{\mathcal{D}}\big(E,\mathbb{Q}(0)\big) & \leftarrow & \mathcal{A}\mathrm{ug}(\mathbb{G}_m) = K^* \otimes \mathbb{Q} \\
\uparrow & & \uparrow & & \\
\mathcal{A}\mathrm{ug}(Y) & \longleftarrow & \mathcal{A}\mathrm{ug}(L^*) & &
\end{array}$$

- Hence, the augmentation space sits in a homotopy pullback square

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- It follows that $\mathcal{A}\mathrm{ug}(L^*)$ has the structure of a $K^* \otimes \mathbb{Q}$-torsor over $\mathcal{A}\mathrm{ug}(Y)$.

Table of Contents

Let P be the pushout

$$\begin{array}{ccc} \operatorname{Sym} C(Y) & \longrightarrow & \operatorname{Sym} E \\ \downarrow & & \downarrow \\ C(Y) & \longrightarrow & P. \end{array}$$

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$$\begin{array}{ccc} \operatorname{Sym} C(Y) & \longrightarrow & \operatorname{Sym} E \\ \downarrow & & \downarrow \\ C(Y) & \longrightarrow & P. \end{array}$$

Then by base-changing everything to $C(Y)$ we find that P is also a pushout

$$\begin{array}{ccc} \operatorname{Sym}_{C(Y)} C(Y) & \longrightarrow & \operatorname{Sym}_{C(Y)} C(L^*) \\ \downarrow & & \downarrow \\ C(Y) & \longrightarrow & P. \end{array}$$

$$\begin{array}{ccc} \mathrm{Sym}_{C(Y)}\, C(Y) & \longrightarrow & \mathrm{Sym}_{C(Y)}\, C(L^*) \\ \downarrow & & \downarrow \\ C(Y) & \longrightarrow & P. \end{array}$$

The copy of $C(L^*)$ in the upper right comes from the Gysin sequence

$$C(Y)(-1)[-2] \xrightarrow{\text{mult. by } c^L} C(Y) \to C(L^*) \to C(Y)(-1)[-1]. \qquad (*)$$

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This shows that P may be identified with the "relatively free commutative algebra" generated by the pointed object

$$C(Y) \to C(L^*)$$

of $\mathrm{Mod}_{C(Y)}$.

Lurie's theory of free algebras shows that

$$P = \operatorname{colim} C(L^*)^{c^L \otimes}$$

is given by the colimit of a diagram

$$C(L^*)^{c^L \otimes} : \mathrm{Fin}_{\mathrm{inj}} \to \mathrm{Mod}_{C(Y)}$$

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which mixes the symmetric group actions on the tensor powers of $C(L^*)$ with multiplication by the 1st Chern class c^L. Using the fact that $\mathrm{Mod}_{C(Y)}$ is tensored over rational spaces, we may decompose the colimit as

$$P = \operatorname{colim}_n \operatorname{Sym}^n_{C(Y)} C(L^*).$$

Taking symmetric powers over $C(Y)$ in the exact triangle

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So finally,

$$P = \operatorname{colim}(C(Y) \to C(L^*) \xrightarrow{=} C(L^*) \xrightarrow{=} C(L^*) \xrightarrow{=} \cdots) = C(L^*).$$

Table of Contents

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$$\mathrm{Aug}^{lg}(X) \to \mathrm{Aug}(A) \xrightarrow{h_L} \mathbb{R}$$

is bounded.

Since $f^*L \simeq \mathcal{O}_X$, f lifts to a map

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$$\tilde{f}\big(X(K_v)\big) \subset \mathcal{L}^*(K_v).$$

Moreover, the local Néron-Tate height

$$\lambda_{L,v} : \mathcal{L}^*(K_v) \to \mathbb{R}$$

is bounded on bounded subsets.

Further, at $v \in \mathcal{Z}$ a place of good reduction, $\lambda_{L,v}$ vanishes on the $\mathcal{O}_v$-lattice

$$\mathcal{L}^*(\mathcal{O}_v) \subset \mathcal{L}^*(K_v).$$

Further, at $v \in \mathcal{Z}$ a place of good reduction, $\lambda_{L,v}$ vanishes on the $\mathcal{O}_v$-lattice

$$\mathcal{L}^*(\mathcal{O}_v) \subset \mathcal{L}^*(K_v).$$

We find that for α ranging over $\mathrm{Aug}^{\mathrm{lg}}(X)$, the sum

$$h_L\big(f(\alpha)\big) = \sum_v \lambda_{L,v}\big(\tilde{f}(\alpha)\big)$$

is a finite sum of bounded functions.