THE PROÉTALE TOPOLOGY ON ADIC SPACES, WITH A VIEW TOWARDS THE HODGE-TATE DECOMPOSITION

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ABSTRACT. These are notes for a talk given in the joint Berkeley-Stanford learning seminar in the Spring of 2024. These should serve as a rapid introduction to the proétale site of an adic space. However, I spend very little space on foundational aspects; instead the focus is more on setting the stage for the application to the Hodge-Tate decomposition, following notes by Bhargav Bhatt from AWS17 [1]. Thus, the main goals, roughly speaking, are to explain the statement that adic spaces over $\mathbb{Q}_p$ are proétale-locally perfectoid, and to give a rough idea of a nontrivial application of this principle.

My understanding is that the literature contains a different site, which was introduced later and given the same name. This talk concerns the version of [7], which serves as our main reference.

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Acknowledgement. Apart from the sources listed above and below, I learned some of this material from Tony Feng, and in particular from his talk in the seminar.

Date: April 12, 2024.

1. Adic spaces

1.1. A topological ring A is *Huber* if it admits an open subring $A_0 \subset A$ which is adic with respect to a finitely generated ideal of definition. A subset $S \subset A$ is *bounded* if for all open neighborhoods U of 0 there exists an open neighborhood V of 0 such that $VS \subset U$. A *pseudo-uniformizer* is a topologically nilpotent unit; a Huber ring is *Tate* if it contains a pseudo-uniformizer; when this is the case, and if $A_0 \subset A$ is any ring of definition and g is any topologically nilpotent unit, then for n sufficiently large, $g^n \in A_0$, and then A_0 is g^n-adic and $A = A_0[(g_0)^{-1}]$.

An element $x \in A$ is *power-bounded* if $\{x^n \mid n \in \mathbb{N}\}$ is bounded; $A^\circ \subset A$ denotes the subring of power-bounded elements. A subring $A^+ \subset A^\circ$ is a *ring of integral elements* if it is open and integrally closed in A. A *Huber pair* is a pair (A, A^+) where A is Huber and $A^+ \subset A$ is a ring of integral elements. A *Tate-Huber pair* or *affinoid pair* is a Huber pair in which A is Tate; see, for instance, Definition 2.6 of [7]. In a similar way, given an affinoid pair (A, A^+), we may speak of an *affinoid* (A, A^+)*-algebra*.

1.2. A totally ordered abelian group Γ is required to obey

$$\gamma \leq \gamma' \quad \text{implies} \quad \delta\gamma \leq \delta\gamma'.$$

Similarly for totally ordered commutative monoid. If Γ is a totally ordered group, we endow $\Gamma \cup \{0\}$ with the structure of a totally ordered monoid by declaring

$$0\gamma = 0 \quad \text{and} \quad 0 \leq \gamma$$

for all $\gamma \in \Gamma$. A *continuous valuation* is given by a totally ordered abelian group Γ and a map

$$A \to \Gamma \cup \{0\}$$

such that $|0| = 0$, $|1| = 1$, $|ab| = |a||b|$, $|a + b| \leq \max(|a|, |b|)$, and for all $\gamma \in \Gamma$ lying in the image of $|\cdot|$, the set $\{a \in A \mid |a| < \gamma\}$ is open in A. The *adic spectrum* $\operatorname{Spa}(A, A^+)$ has underlying set the set of equivalence classes of continuous valuations such that $|A^+| \leq 1$. The *topology* is generated by open subsets of the form

$$\{x \mid |f(x)| \leq |g(x)| \neq 0\}$$

with $f, g \in A$. For $s \in A$ and $T \subset A$ a finite subset that generates an open ideal, the associated *rational subset* is given by

$$U\Big(\frac{T}{s}\Big) = \{x \in X \mid |t(x)| \leq |s(x)| \neq 0 \text{ for all } t \in T\}.$$

Let $X = \operatorname{Spa}(A, A^+)$ and let $U = U(\frac{T}{s}) \subset X$ be a rational subset. We set

$$A\langle T/s\rangle = \widehat{A[1/s]}$$

in which the completion is with respect to a certain topology in which t/s is power-bounded for every $t \in T$, and let $A\langle T/s\rangle^+$ be the completion of the integral closure of the image of $A^+[t/s \,|\, t \in T]$ in $A[1/s]$. See §8.1 of Wedhorn's notes [9] or Theorem 3.1.3 of Scholze-Weinstein [8].

Define

$$(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) = (A\langle T/s\rangle, A\langle T/s\rangle^+).$$

The Huber pair (A, A^+) is *sheafy* if $\mathcal{O}_X$ may be extended to a sheaf (by taking inverse limits over rational subsets). In this case, the stalks are local rings and they have naturally induced valuations. The *affinoid adic space* $\mathrm{Spa}(A, A^+)$ consists of the topological space defined above, the structure sheaf $\mathcal{O}_X$, and the induced valuations on the stalks up to equivalence of valuations.

1.3. An *adic space* is a triple $(X, \mathcal{O}_X, \{|\cdot(x)|\}_{x\in X})$ where X is a topological space, $\mathcal{O}_X$ is a sheaf of topological rings, and for each $x \in X$, $|\cdot(x)|$ is an equivalence class of continuous valuations on $\mathcal{O}_{X,x}$, which is locally of the form $\mathrm{Spa}(A, A+)$. A morphism of adic spaces includes the data of commuting squares

$$\begin{array}{ccc} \mathcal{O}_{Y,f(x)} & \longrightarrow & \mathcal{O}_{X,x} \\ \downarrow & & \downarrow \\ \Gamma_{f(x)} \cup \{0\} & \longrightarrow & \Gamma_x \cup \{0\}, \end{array}$$

but (recalling that up to equivalence of valuations we may always assume the value group is generated by the image of the valuation) these are uniquely determined.

Example 1.4. If A is a topological ring, then we let $A\langle t^{\pm}\rangle$ denote the ring of bidirectional power series $\sum_{n\in\mathbb{Z}} a_n t^n$ over A such that $a_n \to 0$ in both directions. Let $D = \mathrm{Spa}(\mathbb{Q}_p\langle t\rangle, \mathbb{Z}_p\langle t\rangle)$. Then

$$\mathbb{T} := U\Big(\frac{1}{t}\Big) = \mathrm{Spa}(\mathbb{Q}_p\langle t^{\pm}\rangle, \mathbb{Z}_p\langle t^{\pm}\rangle).$$

Here, the ring $\mathbb{Q}_p\langle t^{\pm}\rangle$ of "birestricted formal power series" arises as the p-adic completion of the ordinary localization $\mathbb{Q}_p\langle t\rangle[t^{-1}]$. In modern terminology, '$\mathbb{T}$' stands for *torus*. However, as I learned from Coleman, Dwork used to refer to this adic space (or rather to the associated rigid analytic space) as the "unit tire."

1.5. Fibered products $X \times_Z Y$ of adic spaces sometimes exist. When they do, typically the induced map of topological spaces

$$|X \times_Z Y \to |X| \times_{|Z|} |Y|$$

is surjective with finite fibers.

1.6. If X is affinoid, then $H^i(X, O_X) = 0$ for $i > 0$.

2. Perfectoid spaces

2.1. A Huber ring is *uniform* if A° is bounded. A *perfectoid Tate ring* is a complete uniform Tate ring which admits a pseudo-uniformizer ϖ such that

$$p \in \varpi^p R^\circ$$

(which becomes automatic in characteristic p) and such that the pth power Frobenius map

$$\Phi : R^\circ \to R^\circ/\varpi^p$$

is surjective.[1] The *tilt* of a perfectoid Tate ring is the topological multiplicative monoid given by

$$R^\flat = \varprojlim_{x \mapsto x^p} R$$

with addition defined by

$$(x^{(0)}, x^{(1)}, \dots) + (y^{(0)}, y^{(1)}, \dots) = (z^{(0)}, z^{(1)}, \dots)$$

where

$$z^{(i)} = \lim_{n\to\infty} (x^{(i+n)} + y^{(i+n)})^{p^n}.$$

Let's call a Huber-Tate pair (R, R^+) *perfectoid* just in case R is a perfectoid Tate ring. The *tilt* of a perfectoid Tate-Huber pair (R, R^+) is given by

$$(R, R^+)^\flat = (R^\flat, R^{\flat +}).$$

Example 2.2. Let $\mathbb{Q}_p^{\text{cycl}}$ be the completion of $\mathbb{Q}_p(\mu_{p^\infty})$. Then $\mathbb{Q}_p^{\text{cycl}}$ is perfectoid with tilt

$$(\mathbb{Q}_p^{\text{cycl}})^\flat = \mathbb{F}_p((t^{1/p^\infty}))$$

where

$$t = (1, \zeta_p, \zeta_{p^2}, \dots) - 1$$

for a compatible system of primitive p-power roots of unity. The Galois group

$$\mathbb{Z}_p^\times = \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$$

acts on $\mathbb{F}_p((t^{1/p^\infty}))$ via

$$t \mapsto (1+t)^\gamma - 1.$$

We record some basic facts about perfectoid Tate-Huber pairs; we will not have occasion to make explicit use of these in the sequel.

Lemma 2.3. Let (R, R^+) be a perfectoid Tate-Huber pair.

[1] In fact, this is implied by the apparently weaker condition that the composition

$$R^\circ \xrightarrow{\Phi} R^\circ/\varpi^p \twoheadrightarrow R^\circ/p$$

be surjective.

(1) There exists a pseudo-uniformizer $\varpi \in R$ with $p \in \varpi^p R^\circ$ that admits a sequence of p-power roots, giving rise to an element

$$\varpi^\flat = (\varpi, \varpi^{1/p}, \dots) \in R^{\flat\circ}$$

which is a pseudo-uniformizer of $R^\flat$.

(2) The set of rings of integral elements $R^+ \subset R^\circ$ is in bijection with the set of rings of integral elements $R^{\flat+} \subset R^{\flat\circ}$.

(3) We have

$$R^{\flat+}/\varpi^\flat \xrightarrow{\simeq} R^+/\varpi.$$

2.4. Let's sketch the construction of the map from (3). We have an isomorphism of multiplicative monoids $R^{\flat+} = \varprojlim R^+$. So projection onto the 0th coordinate defines a map of monoids (solid arrows)

$$\begin{tikzcd} R^{\flat+} = \varprojlim S^+ \arrow[r] \arrow[dr, dotted, "f"'] & R^+ \arrow[d] \\ & R^+/\varpi. \end{tikzcd}$$

We claim that the composite f with the projection, as shown, is additive. Indeed

$$\begin{aligned} f((x^{(0)}, x^{(1)}, \dots) + (y^{(0)}, y^{(1)}, \dots)) &= z^{(0)} \\ &= \lim_{n\to\infty} (x^{(n)} + y^{(n)})^{p^n} \\ &\equiv x^{(0)} + y^{(0)} \mod \varpi. \end{aligned}$$

Clearly $f(\varpi^\flat) = \varpi$, which gives us the map. [2]

2.5. A *perfectoid space* is an adic space X covered by affinoid adic spaces U with $\mathcal{O}_X(U)$ perfectoid. The tilts defined above glue to provide global tilts.

Theorem 2.5.1. For any perfectoid space with tilt $X^\flat$, the functor

$$Y \mapsto Y^\flat$$

induces an equivalence of categories between perfectoid spaces over X and perfectoid spaces over $X^\flat$. See, for instance, Theorem 7.1.4 of [SW]. This equivalence restricts to an equivalence of small étale sites.

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[2] In fact, the same construction equally produces a map

$$R^{\flat+}/(\varpi^\flat)^p \to R^+/\varpi^p. \tag{*}$$

3. The étale topology

3.1. According to definition 1.2.1(v) of Huber's book [3], a morphism $f : X \to Y$ of adic spaces is *locally of finite presentation* essentially if in affinoid patches

$$U \to V$$

it's given by morphisms of Huber pairs that are in a suitable sense topologically of finite type: strangely, you impose an additional condition only if $\mathcal{O}(V)$ happens to be discrete, in which case you require

$$\mathcal{O}(U) \leftarrow \mathcal{O}(V)$$

to be of finite presentation in a suitable sense. The morphism f is *étale according to Huber* if it's locally of finite presentation, and for any Huber pair (A, A^+), any square-zero ideal I of A, and any morphism

$$\operatorname{Spa}(A, A^+) \to Y,$$

the map

$$\operatorname{Hom}_Y\left(\operatorname{Spa}(A, A^+), X\right) \to \operatorname{Hom}_Y\left(\operatorname{Spa}(A/I, A^+/I), X\right)$$

is bijective. The morphism f is *finite* if in affinoid patches as above, in addition to a suitable topological finite type condition, both

$$\mathcal{O}(U) \leftarrow \mathcal{O}(V) \quad \text{and} \quad \mathcal{O}^+(U) \leftarrow \mathcal{O}^+(V)$$

are integral ring extensions.

3.2. Scholze, in his thesis [6], defines étale morphisms differently. Let's put ourselves in his setting by fixing a base field k equipped with a nontrivial valuation with values in $\mathbb{R}_{\geq 0}$. Set $k^+ = k^\circ$. A map of affinoid (k, k^+)-algebras

$$(R, R^+) \to (S, S^+)$$

is *finite étale according to Scholze's thesis* if $R \to S$ is finite étale in the sense of algebraic geometry, S has the induced topology, and S^+ is equal to the integral closure of A^+ in S. A morphism of adic spaces

$$f : X \to Y$$

of adic spaces over $\operatorname{Spa}(k, k^+)$ is *finite étale according to Scholze's thesis* if there exists an open affinoid cover $Y = \bigcup Y_i$ such that each $X_i = f^{-1}Y_i$ is affinoid, and each induced map

$$X_i \to Y_i$$

is finite étale in the above sense. Finally, f is *étale according to Scholze's thesis* if for each $x \in X$ there exists an open neighborhood U of x, an open neighborhood V of $f(U)$, and a factorization of the induced map $U \to V$ into an open immersion followed by a finite étale map according to Scholze's thesis. Scholze comments that these definitions allow one to restrict attention to perfectoid spaces despite those being always reduced. Perfectoid spaces, it is said, are rarely locally Noetherian. On the other hand, if we do restrict attention to locally Noetherian adic spaces, then apparently the two definitions coincide. Here we will make only very brief use of the étale site of a non-Noetherian adic space. In any case, below we'll simply say "étale" and leave it for another day to adjudicate between the two definitions.

3.3. If X is an adic space, then étale opens and surjective étale maps form a site. We denote the étale site by $X_{\text{ét}}$. We denote the topos associated to a site by decorating with a tilde.

3.4. Basic aspects of the theory of the étale fundamental group may be imported into the adic setting without change. The finite surjective étale maps form a Galois category $\mathrm{Cov}(X)$. There are typically "geometric points"

$$x : S = \mathrm{Spa}(K, K^+) \to X$$

such that $\tilde{S}_{\text{ét}} \simeq \mathbf{Set}$. Geometric points give rise to fiber functors

$$\omega_x : \mathrm{Cov}(X) \to \mathbf{FinSet}.$$

These, in turn, give rise to profinite groups

$$\pi_1^{\text{ét}}(X, x) = \mathrm{Aut}\,(\omega_x).$$

If Λ is an Artin ring, then there's an equivalence of categories

$$\mathbb{L} \mapsto \mathbb{L}_x$$

between locally constant sheaves of finite Λ-modules and continuous π_1 -representations in finite Λ-modules; the equivalence interchanges the global section functor and the invariants functor.

3.5. Let us sketch a first application of perfectoid methods to establish an important basic property of the étale cohomology of adic spaces. This material will not be used in the sequel.

Theorem (4.9 of [7])**.** Let X be a connected affinoid noetherian adic space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. Fix a geometric base-point x. Let $\mathbb{L}$ be a locally constant sheaf of $\mathbb{F}_p$-vector spaces. Then the map

$$(*) \qquad H^i_{\mathrm{cont}}(\pi_1^{\text{ét}}(X, x), \mathbb{L}_x) \xrightarrow{f^*} H^i(X_{\text{ét}}, \mathbb{L}).$$

is iso.

Sketch. Spaces that are finite étale over X form a site $X_{f\text{ét}}$. One interprets f as the map of topoi

$$\tilde{X}_{\text{fét}} \xleftarrow{f} \tilde{X}_{\text{ét}}$$

induced by the evident inclusion of sites $X_{\text{fét}} \subset X_{\text{ét}}$. Using the Leray spectral sequence, one is reduced to showing that $R^i f_* \mathbb{L} = 0$ for $i > 0$. Since $R^i f_* \mathbb{L}$ is the sheaf associated to the presheaf

$$U \mapsto H^i(U_{\text{ét}}, \mathbb{L}),$$

it's now enough to show that any section of this presheaf is locally in the finite-étale topology zero. Renaming $U = X$, it's enough to show that any cohomology class in $H^i(X_{\text{ét}}, \mathbb{L})$ maps to zero in a finite étale cover. At this point, Scholze imports a construction due to Colmez which provides a perfectoid space X_∞ given (in a suitable sense) as a limit of finite étale covers $X_i \to X$ such that X_∞ has no nontrivial finite étale covers; with its help, we're reduced to showing that $H^i(X_{\infty,\text{ét}}, \mathbb{L}) = 0$ for $i > 0$. We now use the equivalence of étale sites

$$X_{\infty,\text{ét}} \simeq X^\flat_{\infty,\text{ét}},$$

the Artin-Schreier exact sequence

$$0 \to \mathbb{F}_p \to \mathcal{O}_{X^\flat_\infty} \to \mathcal{O}_{X^\flat_\infty} \to 0,$$

and the vanishing $H^i(Y, \mathcal{O}_Y) = 0$ ($i > 0$) for affinoid Y. □

4. The proétale topology

4.1. Let C be a category and recall that colimits in $\mathrm{Fun}(C, \mathrm{Set})$ are computed object-wise, meaning that for each $X \in C$, the associated evaluation functor

$$\mathrm{Fun}(C, \mathrm{Set}) \to \mathrm{Set}$$

preserves colimits. Of course, colimits in $\mathrm{Fun}(C, \mathrm{Set})$ may be identified with limits in $\mathrm{Fun}(C, \mathrm{Set})^{\mathrm{op}}$.

By contrast, the *contravariant Yoneda embedding*

$$y : C \to \mathrm{Fun}(C, \mathrm{Set})^{\mathrm{op}}$$

does *not* in general preserve limits. We refer to objects of $\mathrm{Fun}(C, \mathrm{Set})$ as *copresheaves*. A category I is said to be *cofiltered* if it's nonempty, if given objects i, j there exists a span $i \leftarrow k \to j$, and if given parallel arrows

$$i \rightrightarrows j$$

there exists a morphism $k \to i$ which equalizes the two arrows.

The *pro-category* $\mathrm{Pro}(C)$ is the full subcategory of $\mathrm{Fun}(C, \mathrm{Set})^{\mathrm{op}}$ spanned by those copresheaves that are cofiltered limits of corepresentable copresheaves.

A calculation involving the Yoneda lemma and little else shows that if $F, G \in \mathrm{Pro}(C)$ are given as limits of corepresentable copresheaves as

$$F = \lim_{i \in I} F(i) \quad \text{and} \quad G = \lim_{j \in J} G(j),$$

then

$$(*) \qquad \mathrm{Hom}(F, G) = \lim_{j \in J} \mathrm{colim}_{i \in I} \mathrm{Hom}\,(F(i), G(j)).$$

There's a categorical universal mapping property which we will not use explicitly. More important for us is that $\mathrm{Pro}(C)$ is itself closed under cofiltered limits. For the construction, one shows that if F is a limit of objects in $\mathrm{Pro}(C)$ computed in the category $\mathrm{Fun}(C, \mathrm{Set})^{\mathrm{op}}$ of copresheaves, then the undercategory $C_{F/}$ is cofiltered and F is the limit of the forgetful functor

$$C_{F/} \to \mathrm{Fun}(C, \mathrm{Set})^{\mathrm{op}}.$$

See Chapter 6 of Kashiwara-Shapira [4].

4.2. If C admits finite limits, then so does $\mathrm{Pro}(C)$, and the embedding $p : C \hookrightarrow \mathrm{Pro}(C)$ preserves finite limits. For the latter statement, one considers a finite limit diagram $C = \lim C_i$ in C and an arbitrary test object $T \in \mathrm{Pro}(C)$ and computes $\mathrm{Hom}_{\mathrm{Pro}(C)}(T, C)$ using the fact that filtered colimits commute with finite limits in the category of sets. See lecture notes by Jacob Lurie [5].

Remark 4.3. By contrast, the embedding $y : C \to \mathrm{Fun}(C, \mathrm{Set})^{\mathrm{op}}$ rarely preserves pullbacks. Indeed, if y preserves the pullback square

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z, \end{tikzcd}$$

then for every $T \in C$, the square

$$\begin{tikzcd} \mathrm{Hom}(P, T) & \mathrm{Hom}(Y, T) \arrow[l] \\ \mathrm{Hom}(X, T) \arrow[u] & \mathrm{Hom}(Z, T) \arrow[l] \arrow[u] \end{tikzcd}$$

is a pushout square in the category of sets. So for a concrete counterexample (of some relevance to us), let C be the category of topological spaces, let X, Y be proper open subspaces of a topological space Z, and let $T = P$ be the intersection.

4.4. Let X be an adic space. Facts mentioned below without proof may only hold under some noetherianity assumptions. An object of $\mathrm{Pro}(X_{\text{ét}})$ might be called a "pro-(étale open)" of X. Since later we'll want to distinguish some of these and to call them "pro-étale opens" of X, let's refer to the former as an *étale tower* (or simply a *tower*).

Given towers Y, Z as above, for every $j \in J$ there's evidently a map

$$q_j : \mathrm{Hom}(Y, Z) \to \mathrm{colim}_{i \in I} \mathrm{Hom}\,(Y(i), Z(j)).$$

Fix j. Then for any $f : Y \to Z$, $q_j(f)$ may be represented by an actual morphism

$$f_{i,j} : Y(i) \to Z(j)$$

for any i sufficiently large. Moreover, given any $j' > j$, there exists an $i' > i$ and a commuting square

$$\begin{tikzcd} Y(i') \arrow[r, "f_{i',j'}"] \arrow[d] & Z(j') \arrow[d] \\ Y(i) \arrow[r, "f_{i,j}"'] & Z(j). \end{tikzcd}$$

In this case, let's say that $f_{i',j'}$ *dominates* $f_{i,j}$. I believe a morphism f as above should be considered *surjective* if every associated morphism $f_{i,j}$ of ordinary étale opens is dominated by a surjective morphism.

A morphism

$$U \to V$$

of towers is *étale* if it arises as a pullback of an étale morphism of (ordinary) étale opens of X. Similarly for *finite étale*.

A morphism of towers as above is *proétale* if U may be written as a filtered limit

$$U = \lim U_i \to V$$

of towers U_i over V (that is, a limit in the overcategory $(\mathrm{Pro}\, X_{\text{ét}})_{/V}$), with each

$$U_i \to V$$

an étale morphism of towers, and such that for all $i > j$ sufficiently large, the morphism of towers $U_i \to U_j$ is finite étale and surjective.

4.5. If U is a tower, we define the *underlying topological space* by

$$|U| = \lim |U_i|.$$

One can see this is well defined as follows. The category of cofiltered diagrams $F : I \to C$ in a category C with homs defined by the formula 4.1(*)

maps equivalently to Pro C. Given diagrams $U : I \to X_{\text{ét}}$, $V : J \to X_{\text{ét}}$, there are natural maps

$$\begin{array}{c}
\lim_j \operatorname{colim}_i \operatorname{Hom}(U_i, V_j) \\
\downarrow \\
\lim_j \operatorname{colim}_i \operatorname{Hom}(|U_i|, |V_j|) \\
\downarrow \\
\lim_j \operatorname{Hom}(\lim_i |U_i|, |V_j|) \\
\| \\
\operatorname{Hom}(\lim_i |U_i|, \lim_j |V_j|)
\end{array}$$

which make $U \mapsto |U|$ into a functor on the category of diagrams.

4.6. We define the *proétale site* $X_{\text{proét}}$ as follows. Objects (which we'll refer to as *proétale opens of* X) are those étale towers over X which are proétale over X. Morphisms are just morphisms of towers. A covering is given by a family of proétale morphisms

$$\{f_i : U_i \to U\}$$

such that the maps

$$|U_i| \to |U|$$

of underlying topological spaces are jointly surjective.

Remark 4.7. The objects of the proétale site may be made somewhat more concrete as follows. Suppose the tower $U \in \operatorname{Pro}(X_{\text{ét}})$ belongs to proét(X) and write $U = \lim U_i \to X$ as above, where now X plays the role of V. The requirement that $U_i \to X$ be étale means that we have a pullback square

$$\begin{tikzcd}
U_i \arrow[r, "f"] \arrow[d] & X \arrow[d, "g'"] \\
U_i' \arrow[r, "f'"'] & X'
\end{tikzcd}$$

in $\operatorname{Pro}(X_{\text{ét}})$ in which f' is an étale morphism in $X_{\text{ét}}$. In particular, X' belongs to $X_{\text{ét}}$ and so g' is a section of an étale cover of X, hence (at least under mild assumptions) an isomorphism onto a union of components. It follows that U_i is just an étale open of X. Thus, to summarize, any proétale open of X may be written as a limit (computed in the $\operatorname{Pro}(X_{\text{ét}})$) of a cofiltered diagram of étale opens of X in which the structure maps are eventually finite étale.

Proposition 4.8. This defines a site.

Sketch. We have to show, for instance, that if $f : U \to V$ is a proétale cover of a proétale open, and $W \to V$ is a morphism of proétale opens, then the pullback $U \times_V W$ exists and

$$f_W : U \times_V W \to W$$

is a proétale cover. The existence follows formally from the existence of products of étale opens, and we take this as given. A not purely formal point is the surjectivity of the map of underlying topological spaces. In fact, there's a commuting square of topological spaces

$$\begin{tikzcd} {|U \times_V W|} \arrow[r, "|f_W|"] \arrow[d] & {|W|} \arrow[d] \\ {|U|} \arrow[r] & {|V|} \end{tikzcd}$$

hence a map

$$(*) \qquad |U \times_V W| \to |U| \times_{|V|} |W|$$

factoring $|f_W|$, and it's enough to show that (*) is surjective. (Case 0) When $U, V, W \in X_{ét}$ are ordinary étale opens of X, it's a general fact that (*) is not only surjective, but also has finite fibers. (Case 1) Suppose next that $U, V \in X_{ét}$ but $W \in X_{proét}$. Write $W = \lim W_i$ with $W_i \in X_{ét}$, the limit being computed in $\mathrm{Pro}(X_{ét})$. We have, by definition,

$$(A) \qquad |U \times_V W| \simeq \lim |V \times_V W_i|.$$

We also have

$$(B) \qquad \lim |U| \times_{|V|} |W_i| \simeq |U| \times_{|V|} |W|$$

because fiber products commute with inverse limits. We claim that the map $A \to B$ has nonempty compact fibers. In deed, this is a general fact about topological spaces as follows. Let $\{f_i : A_i \to B_i\}_i$ be a map of filtered limit diagrams of topological spaces such that each f_i is surjective with compact fibers and fix $b \to B$ a point. Then

$$A \times_B b = \lim(A_i \times_{B_i} b)$$

is a limit of nonempty compact spaces, hence (by an application of Tychonoff's theorem) compact and nonempty. (Case 3) Turning to the general case with $U, V, W \in X_{proét}$ proétale opens, we fix a proétale presentation

$$U = \lim_j U_j \to V$$

and apply the same argument as in case (2), this time taking limits over j. $\square$

4.9. Let K be a perfectoid field of characteristic 0 with an open and bounded valuation subring $K^+ \subset K$, and let X be an adic space over $\mathrm{Spa}(K, K^+)$. Some of the assertions below may depend on a suitable noetherianity condition. A finite étale tower $U = \{U_i\}_{i \in I} \in X_{\text{proét}}$ is *affinoid perfectoid* if it satisfies the following two conditions. (1) We require each U_i to be affinoid, $= \mathrm{Spa}(R_i, R_i^+)$. In terms of the rings R_i, R_i^+, we define

$$R^+ = \lim_n (\mathrm{colim}_i R_i^+)/p^n \quad \text{and} \quad R = R^+[p^{-1}].$$

(2) We require that (R, R^+) be a perfectoid Tate-Huber pair.

As a matter of notation, we set

$$\hat{U} = \mathrm{Spa}(R, R^+).$$

Theorem 4.10. Let X be a locally noetherian adic space over $\mathrm{Spa}(K, K^+)$. Then every $U \in X_{\text{proét}}$ admits a proétale cover by affinoid perfectoid spaces.

Example 4.11. Let

$$\mathbb{T} = \mathbb{T}_{(K,K^+)} = \mathrm{Spa}(K\langle t^{\pm}\rangle, K^+\langle t^{\pm}\rangle).$$

We re-spell the Huber pair appearing above in terms of the rings

$$(K\langle t^{\pm 1/p^m}\rangle, K^+\langle t^{\pm 1/p^m}\rangle)$$

of birestricted formal power series in t^{1/p^m}. This implies inclusions

$$(K\langle t^{\pm 1/p^m}\rangle, K^+\langle t^{\pm 1/p^m}\rangle) \subset (K\langle t^{\pm 1/p^{m+1}}\rangle, K^+\langle t^{\pm 1/p^{m+1}}\rangle).$$

The resulting tower

$$\tilde{\mathbb{T}} = (\cdots \mathbb{T} \to \mathbb{T} \to \mathbb{T})$$

forms an affinoid perfectoid proétale cover of $\mathbb{T}$. The associated affinoid perfectoid space is denoted by

$$\hat{\mathbb{T}} = \mathrm{Spa}(K\langle t^{\pm 1/p^\infty}\rangle, K^+\langle t^{\pm 1/p^\infty}\rangle).$$

The elements of the ring $K\langle t^{\pm 1/p^\infty}\rangle$ may be represented by formal power series

$$\sum_{i \in \mathbb{Z}} c_i t^{d_i}$$

with $d_i \in \mathbb{Z}[1/p]$ and $c_i \in K$ such that

$$c_i \to 0 \quad \text{as} \quad i \to \infty,$$

and this is helpful when verifying that $\hat{\mathbb{T}}$ is indeed perfectoid.

The same in several variables defines an explicit proétale cover

$$\tilde{\mathbb{T}}^n \to \mathbb{T}^n$$

of the n-fold adic torus.

Sketch of proof of Theorem 4.10. Fix $U \in X_{\text{proét}}$ an affinoid perfectoid. We restrict attention to the case that X is smooth over (K, K^+). Under this assumption, Huber [3, Corollary 1.6.10] shows that X admits an open cover by adic subspaces that admit an étale map to a perfectoid torus $\mathbb{T}^n$. Fix such an open X' and an étale map

$$X' \to \mathbb{T}^n.$$

Let $U' \to X'$ be the pullback of U, which is now proétale over X' (hence also proétale over X). The fiber product

$$U'' = U' \times_{\mathbb{T}^n} \tilde{\mathbb{T}}^n$$

exists, and is then proétale both over U' and over $\tilde{\mathbb{T}}^n$. Scholze [7, Lemma 4.6] proves that any proétale cover of a perfectoid proétale open (in a suitable sense) is again perfectoid, and derives from this that U'' may be covered by affinoid perfectoids.[3] □

5. Completed structure sheaves on the proétale site

5.1. Let X be a locally noetherian adic space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. The evident functor between sites

$$X_{\text{proét}} \leftarrow X_{\text{ét}}$$

induces a morphism of topoi

$$\nu : \tilde{X}_{\text{proét}} \to \tilde{X}_{\text{ét}}.$$

We set

$$\begin{aligned} O_X^{+,\text{proét}} &:= \nu^* O^+_{X_{\text{ét}}}, \\ \hat{O}_X^{+,\text{proét}} &:= \lim_n O_X^{+,\text{proét}}/p^n, \\ \hat{O}_X^{\text{proét}} &:= \hat{O}_X^{+,\text{proét}}[1/p]. \end{aligned}$$

5.2. Let C be a complete algebraically closed nonarchimedean field containing $\mathbb{Q}_p$ and let X be a smooth proper adic space over C. Then the Leray spectral sequence associated to ν and to $\hat{O}_X^{\text{proét}}$ has terms

$$E_2^{i,j} = H^i(X, \Omega^j_{X/C})(-j) \Rightarrow H^{i+j}_{\text{ét}}(X, C).$$

That is, we have, on the one hand,

$$H^i(X_{\text{proét}}, \hat{O}_X) \simeq H^i_{\text{ét}}(X, C),$$

and on the other hand

$$R^j \nu_* \hat{O}_X^{\text{proét}} \simeq \Omega^j_{X/C}(-j).$$

[3]Technically speaking, this involves a notion of analytic open covers of proétale opens which we do not go into.

Our goal for the rest of this lecture is to set the stage for the construction of this last isomorphism.

Theorem 5.3 (Lemma 4.10 of [7])**.** Let K be a perfectoid field of characteristic 0 with an open and bounded valuation subring K^+ as in paragraph 4.9, let X be a locally noetherian adic space over $\mathrm{Spa}(K, K^+)$, and let $U \in X_{\text{proét}}$ be an affinoid perfectoid proétale open with associated perfectoid space

$$\hat{U} = \mathrm{Spa}(R, R^+).$$

Then

$$\hat{\mathcal{O}}_X^{+,\text{proét}}(U) = R^+, \quad \text{and} \tag{1}$$

$$H^i(U, \hat{\mathcal{O}}_X^{\text{proét}}) = 0 \quad \text{for} \quad i > 0. \tag{2}$$

Example 5.4. We do not discuss the proof of Theorem 5.3. Instead, let us describe a typical application. Following Bhatt [1], we work over the base $\mathrm{Spa}(C, \mathcal{O}_C)$ were C is the completion of an algebraic closure of $\mathbb{Q}_p$. Then using Theorem 5.3 we can construct an isomorphism

$$H^i(\mathbb{T}, \hat{\mathcal{O}}_{\mathbb{T}}^{\text{proét}}) = H^i_{\text{continuous}}(\mathbb{Z}_p(1), C\langle T^{\pm 1/p^\infty}\rangle)$$

($i > 0$). The construction, in outline, goes as follows. The unit tire written in the form

$$\mathbb{T}_i = \mathrm{Spa}\,(C\langle T^{\pm 1/p^i}\rangle, \mathcal{O}_C\langle T^{\pm 1/p^i}\rangle)$$

is in a natural way an étale μ_{p^i}-torsor over $\mathbb{T} = \mathrm{Spa}\,(C\langle T^{\pm}\rangle, \mathcal{O}_C\langle T^{\pm}\rangle)$. Let $S = \mathrm{Spa}(C, \mathcal{O}_C)$. We define $\underline{\mathbb{Z}}_p(1)$ to be the object of $S_{\text{proét}}$ given by the tower of (noncanonically constant) étale groups μ_{p^i} over S. Equivalently,

$$\underline{\mathbb{Z}}_p(1) = \lim_i \mu_{p^i}$$

where the limit is computed in $S_{\text{proét}}$. This proétale S-group is a "topologically constant sheaf" in the following sense: given $U \in S_{\text{proét}}$,

$$\underline{\mathbb{Z}}_p(1) = \mathrm{Hom}_{\text{Top.Sp.}}(|U|, \mathbb{Z}_p(1)).$$

On can, in an evident way, base-change $\underline{\mathbb{Z}}_p(1)$ to X, and a similar comment applies over X. The availability of topologically constant sheaves like $\underline{\mathbb{Z}}_p(1)$ is a big part of the charm of the proétale site.

Since the torsor structures are compatible for varying i, the affinoid perfectoid proétale cover

$$\tilde{\mathbb{T}} \to \mathbb{T} \tag{*}$$

is a $\underline{\mathbb{Z}}_p(1)$-torsor. According to Theorem 5.3(2), $H^i(\tilde{\mathbb{T}}, \hat{\mathcal{O}}_{\mathbb{T}}^{\text{proét}}) = 0$. This implies that $H^i(\mathbb{T}, \hat{\mathcal{O}}_{\mathbb{T}}^{\text{proét}})$ may be computed using Cech cohomology with respect to the covering (*), hence from a complex of the form

$$\hat{\mathcal{O}}_{\mathbb{T}}^{\text{proét}}(\tilde{\mathbb{T}}) \to \hat{\mathcal{O}}_{\mathbb{T}}^{\text{proét}}(\tilde{\mathbb{T}} \times_{\mathbb{T}} \tilde{\mathbb{T}}) \to \hat{\mathcal{O}}_{\mathbb{T}}^{\text{proét}}(\tilde{\mathbb{T}} \times_{\mathbb{T}} \tilde{\mathbb{T}} \times_{\mathbb{T}} \tilde{\mathbb{T}}) \to \cdots . \tag{**}$$

According to Theorem 5.3(1),

$$\hat{O}_{\mathbb{T}}^{\text{proét}}(\tilde{\mathbb{T}}) = C\langle T^{\pm 1/p^\infty}\rangle.$$

The torsor structure gives us isomorphisms such as

$$\underline{\mathbb{Z}}_p(1) \times \tilde{\mathbb{T}} \xrightarrow{\sim} \tilde{\mathbb{T}} \times_{\mathbb{T}} \tilde{\mathbb{T}}.$$

Together, we have, for instance

$$\begin{aligned} \hat{O}_{\mathbb{T}}^{\text{proét}}(\tilde{\mathbb{T}} \times_{\mathbb{T}} \tilde{\mathbb{T}}) &\simeq \hat{O}_{\mathbb{T}}^{\text{proét}}(\underline{\mathbb{Z}}_p(1) \times_{\mathbb{T}} \tilde{\mathbb{T}}) \\ &\overset{!}{\simeq} \mathrm{Hom}_{\text{top.sp.}}\left(\mathbb{Z}_p(1), \hat{O}_{\mathbb{T}}^{\text{proét}}(\tilde{\mathbb{T}})\right) \\ &\simeq \mathrm{Hom}_{\text{top.sp.}}\left(\mathbb{Z}_p(1), C\langle T^{\pm 1/p^\infty}\rangle\right) \end{aligned}$$

and similar logic shows that the entire complex (**) is isomorphic to the standard complex computing the continuous group cohomology.

Let us note that the isomorphism marked with a shriek is a continuous analog of the following classical maneuver. In many familiar sites, the coprojections of a coproduct form a covering. Suppose U is an object of such a site C, let $\mathcal{F}$ be a sheaf of sets, and let N be a set. Then

$$\mathcal{F}(\underline{N} \times U) = \mathcal{F}(U^{\coprod N}) = \mathcal{F}(U)^{\prod N} = \mathrm{Hom}_{\text{Set}}\left(N, \mathcal{F}(U)\right).$$

6. The (completed) cotangent complex of the completed proétale sheaf of bounded functions

6.1. If $A \to B$ is a map of rings, we let

$$A \to P^\bullet_{B/A} \to B$$

be the canonical resolution of B by free A-algebras, and we define the ("algebraic") cotangent complex by

$$L_{B/A} = \Omega^1_{P^\bullet_{B/A}} \otimes_{P^\bullet_{B/A}} B,$$

regarded as an object of the derived ∞-category of B-modules. If $X \to \mathrm{Spa}(C, O_C)$ is an adic space, we define $L_{\hat{O}_X^{+,\text{proét}}/O_C}$ by a fancy sheafification (in the proétale topology) of the presheaf

$$U \mapsto L_{\hat{O}_X^{+,\text{proét}}(U)/O_C}.$$

Proposition 6.2. We have

$$L_{\hat{O}_X^{+,\text{proét}}/O_C} \otimes^L_{\mathbb{Z}_p} \mathbb{F}_p \simeq 0.$$

Sketch. I'll only indicate how some of the theorems and propositions mentioned above are relevant. We use theorem 4.10 to reduce to showing that

$$L_{\hat{O}_X^{+,\text{proét}}(U)/O_C} \otimes^L_{\mathbb{Z}_p} \mathbb{F}_p \simeq 0$$

for affinoid perfectoid $U \in X_{\text{proét}}$. By Theorem 5.3(1), $R^+ = \hat{O}_X^{+,\text{proét}}(U)$ is perfectoid. By a compatibility of cotangent complexes with derived base-change, we're now reduced to showing that $L_{(R^+/p)/(O_C/p)} = 0$. Now one shows that on the one hand, the relative Frobenius

$$(*) \qquad F_{(R^+/p)/(O_C/p)} : (R^+/p)^{(1)} \to R^+/p$$

of R^+/p over O_C/p is zero. Here, while the surjectivity follows directly from the definition of "perfectoid", the injectivity is less clear.

On the other hand, relative Frobenius induces zero on cotangent complexes. Indeed, straight from the definition one is reduced to considering the relative differentials of a polynomial algebra, where the vanishing follows from the fact that $d(t^p) = 0$. Together, the map of cotangent complexes induced by relative Frobenius is both zero and an isomorphism, so its source and target are zero. □

6.3. It may seem strange that we are here interested in the algebraic cotangent complex of a morphism of sheaves of topological rings. My understanding is that we make up for ignoring the topology by considering only a small portion of $L_{\hat{O}_X^{+,\text{proét}}(U)/O_C}$. Indeed, above we only considered the quotient by p; below we'll consider the *derived p-adic completion* $\hat{L}_{\hat{O}_X^{+,\text{proét}}/O_C}$, whose vanishing apparently follows from the vanishing modulo p. For the notion of derived completion, and its relationship to the *analytic* cotangent complex, see Chapter 7 of Gabber-Ramero arXiv:math/0201175v3 [2].

7. Appendix: Calculation of the derived pushforward sheaves

Above we introduced the proétale site of an adic space, we explained what it means to say that (in some level of generality) adic spaces are proétale locally perfectoid, and we demonstrated the use of this principle in two different sorts of applications. In one direction, we showed in a concrete example (5.4) how the cohomology of the completed proétale structure sheaf may be computed as a continuous group cohomology. In another direction, we explained how to go in the direction of the vanishing of the derived completion of the relative cotangent complex of the completed proétale sheaf of bounded functions (6.3).

This last section forms a kind of appendix in which we indicate briefly how these two applications contribute to the calculation of the derived pushforward sheaves $R^j\nu_*\hat{O}_X^{\text{proét}}$ indicated in paragraph 5.2. As we'll explain (or rather, as we'll merely hint), the vanishing of the derived completion of the cotangent complex contributes to the computation of the first derived pushforward $R^1\nu_*\hat{O}_X^{\text{proét}}$. Meanwhile, continuous group cohomology plays a role in climbing up to higher derived pushforwards. The precise statement (Lemma 3.3.1 of Bhatt [1]) is that $R^1\nu_*\hat{\mathcal{O}}_X$ is locally free of rank n and

$$R^i\nu_*\hat{\mathcal{O}}_X = \wedge^i R^1\nu_*\hat{\mathcal{O}}_X.$$

It's not too hard to see why the interpretation in terms of continuous group cohomology comes into play here, since it gives us concrete complexes to work with.

We'll leave it at that and turn instead to the computation of $R^1\nu_*\hat{\mathcal{O}}_X$. This is an abbreviated account of the already rather abbreviated account in sections 3.3. and 3.4 of Bhatt's AWS notes [1].

Thus, our goal here is to give a rough idea of how the vanishing

$$\text{(V)} \qquad \hat{L}_{\hat{\mathcal{O}}_X^{+,\text{proét}}/\mathcal{O}_C} = 0$$

contributes to the construction of a morphism

$$\Phi^1 : \Omega^1_{X/C}(-1) \to R^1\nu_*\hat{\mathcal{O}}_X^{\text{proét}},$$

where $\Omega^1_{X/C}$ means *analytic* differential forms. In addition to the vanishing (*), a key ingredient in the construction is a theorem due to Fontaine, which, seen through a certain lens, says that $d\log$ induces a close relationship (if not quite an isomorphism)

$$\text{(CR)} \qquad \mathcal{O}_C[1] \dashrightarrow \hat{L}_{\mathcal{O}_C/\mathbb{Z}_p}.$$

We consider the sequence

$$\mathbb{Z}_p \to \mathcal{O}_C \to \hat{\mathcal{O}}_X^{+,\text{proét}}$$

of sheaves of rings on $X_{\text{proét}}$. We apply derived completion to the associated triangle of cotangent complexes:

$$\hat{L}_{\mathcal{O}_C/\mathbb{Z}_p}\hat{\otimes}_{\mathcal{O}_C}\hat{\mathcal{O}}_X^{+,\text{proét}} \xrightarrow{e} \hat{L}_{\hat{\mathcal{O}}_X^{+,\text{proét}}/\mathbb{Z}_p} \to \hat{L}_{\hat{\mathcal{O}}_X^{+,\text{proét}}/\mathcal{O}_C}.$$

In view of the vanishing of the object on the right, the map on the left is an equivalence. Inverting p and applying Fontain's theorem, we obtain an isomorphism

$$(*) \qquad \hat{L}_{\hat{\mathcal{O}}_X^{\text{proét}}/\mathbb{Z}_p} \simeq \hat{\mathcal{O}}_X^{\text{proét}}(1)[1].$$

There's a natural map

$$(**) \qquad \nu^* L_{\mathcal{O}_X/\mathbb{Z}_p} \to \hat{L}_{\hat{\mathcal{O}}_X^{\text{proét}}/\mathbb{Z}_p}.$$

Composing (*) with (**) and using adjunction we obtain a map

$$L_{\mathcal{O}_X/\mathbb{Z}_p} \to R\nu_*\hat{\mathcal{O}}_X^{\text{proét}}(1)[1].$$

Taking H^0, we finally get our map

$$\Omega^1_X \to R^1\nu_*\hat{\mathcal{O}}_X^{\text{proét}}(1).$$

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