

Towards coarse moduli of augmentations

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Outline

- 1 Preview
- 2 Properties of $\mathcal{D}_M$
- 3 From points to augmentations
- 4 The Iwanari tower
- 5 Compatibility with Chabauty-Kim
- 6 Coarse criterion
- 7 Refined criterion

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- K a number field.
- $Z \subset \operatorname{Spec} \mathcal{O}_K$ open.¹
- $\operatorname{Sm}_Z =$ smooth Z -schemes.
- $\mathcal{D}_M = \operatorname{DM}(Z, \mathbb{Q}) = \operatorname{Stab}_{\mathbb{P}^1} \operatorname{Loc}_{\mathbb{A}^1} \operatorname{Sh}_{\text{ét}}(\operatorname{Sm}_Z, \mathcal{D}(\mathbb{Q}))$.
- Preview:
 - ▶ A variety X over Z gives rise to a highly structured algebra $C^*(X)$ in $\mathcal{D}_M$.
 - ▶ A Z -point gives rise to an augmentation $C^*(X) \rightarrow \mathbb{1}$.
 - ▶ After fixing a base-point, the assignment $X(Z) \rightarrow \operatorname{Aug}(C^*(X))$ factors the “unipotent Kummer map” of Chabauty-Kim theory from points to Galois-equivariant torsors under the prounipotent completion of $\pi_1^{\text{ét}}(X_{\overline{\mathbb{Q}}})$.
 - ▶ This suggests the possibility of *performing* Chabauty-Kim theory motivically without waiting for a t-structure.
 - ▶ With this application in mind, we’ll focus on the representability of $\operatorname{Aug}(C^*(X))$.

¹At least in most of our examples.

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- Properties

- ▶ It's a presentably symmetric monoidal stable $\mathbb{Q}$ -linear ∞ -category.
- ▶ Let $\mathcal{C}_M = \text{CAlg } \mathcal{D}_M$. There are functors²

$$\begin{array}{ccc} \text{Sm}_Z & \xrightarrow{C_*} & \mathcal{D}_M \\ & & \\ \text{Sm}_Z^{\text{op}} & \xrightarrow{C^*} & \mathcal{C}_M \\ & \searrow C^* = C_*^\vee & \downarrow \\ & & \mathcal{D}_M. \end{array}$$

- ▶ For $X \in \text{Sm}_Z$, there are isomorphisms

$$\pi_0 \text{Hom}_{\mathcal{D}_M} (C_*(X), \mathbb{Q}(i)(j)) \simeq K_{2i-j}^{(i)}(X).$$

²The functor C_* can equally be upgraded to coalgebras.

- More properties

- ▶ There's a W -realization functor W^* associated to any suitable Weil cohomology theory, e.g. given a complex embedding $K \subset \mathbb{C}$, there's an equivalence

$$\mathbb{C} \otimes_{\mathbb{Q}} \mathcal{R}_{\text{Betti}} C^*(X) \simeq \Omega_{\text{smooth}}^{\bullet}(X(\mathbb{C}))$$

in $\text{CAlg } \mathcal{D}(\mathbb{C})$; or if X is affine, then

$$\mathcal{R}_{\text{dR}} C^*(X) \simeq \Omega_{\text{alg}}^{\bullet}(X_K)$$

in $\text{CAlg } \mathcal{D}(K)$.

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We have $C^*(Z) = \mathbb{1}$ (the tensor unit $\mathbb{Q}(0)$ with its essentially unique algebra structure).

By functoriality, a Z -point $x \in X(Z)$ gives rise to an augmentation

$$C^*(X) \xrightarrow{x} \mathbb{1}.$$

Definition

- Given $C \in \mathcal{C}_M$, we consider an associated presheaf of homotopy types

$$\mathcal{A}ug(C) : \mathbf{Aff}_{\mathbb{Q}}^{\mathrm{op}} \rightarrow \mathcal{H}$$

with values

$$\mathcal{A}ug(C)(R) = \mathrm{Hom}_{\mathcal{C}_M}(C, R \otimes \mathbb{1})$$

on any $\mathbb{Q}$ -algebra R .

- For $X \in \mathrm{Sm}_{\mathbb{Z}}$, we let

$$\mathcal{A}ug(X) := \mathcal{A}ug(C^*(X)).$$

Question

When is the presheaf of connected components $\pi_0 \mathcal{A}ug(X)$ representable by a finite type $\mathbb{Q}$ -variety.

Answer: Sometimes this is literally the case. For instance,

Theorem (Follows from work of Ancona et. al.)

Let $A \rightarrow Z$ be a semiabelian scheme. Then ^a

$$\mathcal{A}ug(A) \simeq \mathbb{V}^{\vee} A(Z)_{\mathbb{Q}}.$$

$$^a\mathbb{V}^{\vee}(-) = \text{Spec Sym}(-)^{\vee}.$$

And also,

Theorem A (Direct consequence of joint work with L. A. Betts)

Let $L^* \rightarrow A \rightarrow Z$ be a $\mathbb{G}_m$ -bundle over a semiabelian scheme. Then the augmentation space is a trivial torsor:

$$\begin{array}{ccc} \mathbb{V}^{\vee} \mathbb{G}_m(Z)_{\mathbb{Q}} & \begin{array}{c} \circlearrowright \end{array} & \mathcal{A}ug(L^*) \\ & & \downarrow \\ & & \mathbb{V}^{\vee} A(Z)_{\mathbb{Q}}. \end{array}$$

Remark

Theorem A is based on a determination of the structure of $C^*(L^*)$. The latter instantiates the expectation that highly structured algebras in motives with rational coefficients should form a reasonably computable flavor of unstable motivic homotopy.

Proof later, time permitting.

Answer: But usually probably very infinitary.

Solution: Use a filtration / tower studied by I. Iwanari.

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Given $C \in \mathcal{C}_M$, define the tower on the left...

$$\begin{array}{c}
 C \\
 \uparrow \\
 \vdots \\
 \uparrow \\
 \mathfrak{I}_2 C \longleftarrow \mathrm{Sym} V_2 \\
 \uparrow \\
 \mathfrak{I}_1 C \longleftarrow \mathrm{Sym} V_1 \\
 \uparrow \\
 \mathbb{1} \longleftarrow \mathrm{Sym} V_0
 \end{array}$$

$$V_n \longrightarrow \mathfrak{I}_n C \longrightarrow C \longrightarrow V_n[1]$$

$$\begin{array}{ccc}
 \mathfrak{I}_{n+1} C & \longleftarrow & \mathbb{1} \\
 \uparrow & & \uparrow \\
 \mathfrak{I}_n C & \longleftarrow & \mathrm{Sym} V_n
 \end{array}$$

by the exact triangles in $\mathcal{D}_M$ and the pushout squares in $\mathcal{C}_M$ on the right.

Definition

- $\mathcal{A}ug^n(C) = \mathcal{A}ug(\mathfrak{I}_n C)$
- $\mathcal{A}ug^n(X) = \mathcal{A}ug(\mathfrak{I}_n C^*(X))$.

Variant

- $\mathcal{A}ug_W^n(X) = \mathcal{A}ug(W^* \mathfrak{I}_n C^*(X))$.

Outline of talk:

- Compatibility with Chabauty-Kim
- Coarse criterion for representability
- Example: X a punctured elliptic scheme, $n = 3$.
- Refined criterion

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Assume $X \rightarrow Z$ is the complement of an étale divisor in a smooth proper scheme.

Fix a Z -integral base-point x .

$\mathrm{Loc}^{\mathrm{un}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_p) =$ unipotent p -adic local systems.

Let $\pi_1^{\mathrm{un}}(X, x)_{\mathbb{Q}_p}^{\mathrm{\acute{e}t}} = \pi_1(\mathrm{Loc}^{\mathrm{un}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_p), \omega_x)$.

$\mathfrak{R}_{\mathbb{Q}_p}^{\mathrm{\acute{e}t}} = p$ -adic étale realization.

Theorem (Long history, including joint work with T. Schlank, joint work with L. A. Betts, plus work of Tur-Dorvault in the case of tangential base points)

There's a $G_{\mathbb{Q}}$ -equivariant isomorphism

$$\mathrm{Spec} H^0 \mathfrak{R}_{\mathbb{Q}_p}^{\mathrm{\acute{e}t}} \mathrm{Bar}(C^*(X) \xrightarrow{x} \mathbb{1}) = \pi_1^{\mathrm{un}}(X, x)_{\mathbb{Q}_p}^{\mathrm{\acute{e}t}}.$$

Similar result for path torsors in place of fundamental groups. Similar result in the mixed Hodge realization.

Definition

The ***unipotent fundamental group*** is the family of prounipotent groups

$$\pi_1^{\text{un}}(X, x)^W = \text{Spec } H^0 \mathfrak{R}^W \text{Bar}(C^*(X) \xrightarrow{x} \mathbb{1})$$

indexed by the various Weil cohomology theories W , together with extra structures native to W (Hodge structure, Galois action, crystalline Frobenius...).

Remark:

If we ignore the extra structures, then this follows from “cellular Tannaka duality”:

Let $\text{Cell}(X_{\overline{\mathbb{Q}}}) =$ complexes whose cohomology sheaves are ind-unipotent local systems.

Let $C_{\text{ét}}^*(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_p) := \mathfrak{R}_{\mathbb{Q}_p}^{\text{ét}} C^*(X) \in \text{CAlg } \mathcal{D}(\mathbb{Q}_p)$.

Let $B_x^{\mathbb{Q}_p\text{-ét}} := \mathfrak{R}_{\mathbb{Q}_p}^{\text{ét}} \text{Bar}(C^*(X) \xrightarrow{x} \mathbb{1}) \in \text{coAlg CAlg } \mathcal{D}(\mathbb{Q}_p)$.

Then there are equivalences

$$\text{Cell}(X_{\overline{\mathbb{Q}}}) \simeq \mathcal{D} \text{ Mod}_{C_{\text{ét}}^*(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_p)}(\mathcal{D}(\mathbb{Q}_p)) \simeq \mathcal{D}^{\text{co}} \text{ coMod}_{B_x^{\mathbb{Q}_p\text{-ét}}}(\text{CAlg } \mathcal{D}(\mathbb{Q}_p)).$$

The Betti version is similar with $C_{\text{Sing}}^*(X(\mathbb{C}), \mathbb{Q})$ and cellular sheaves of $\mathbb{Q}$ -vector spaces on $X(\mathbb{C})$.

Example: $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$.

- $C^*(X) = \mathbb{Q}(0) \oplus \mathbb{Q}(-1)[-1]^{\oplus 2}$.
- The multiplication is stupid.
- The underlying morphism

$$\mathbb{Q}(0) \oplus \mathbb{Q}(-1)[-1]^{\oplus 2} \rightarrow \mathbb{Q}(0)$$

in $\mathcal{D}_M$ induced by an augmentation contains little information (e.g. no information if $Z = \operatorname{Spec} \mathbb{Z}$).

- But the higher coherences remember the entire unipotent fundamental group and its path torsors.
- The latter has many nontrivial periods (e.g. already for $Z = \operatorname{Spec} \mathbb{Z}$, all multiple zeta values).

Theorem B (Compatibility with CK)

For simplicity, let $Z \subset \operatorname{Spec} \mathbb{Z}$ ($K = \mathbb{Q}$). Fix $p \in Z$. Fix n and let $\pi_1^{[n]}(X, x)$ denote the n th quotient along the dcs. Then there's a commutative diagram

$$\begin{array}{ccc}
 X(Z) & \xrightarrow{\quad} & X(\mathbb{Z}_p) \\
 \downarrow & & \downarrow \\
 \pi_0 \operatorname{Aug}^n(X)_{\mathbb{Q}_p} & \xrightarrow{\quad} & \pi_0 \operatorname{Aug}_{F\phi}^n(X_{\mathbb{Q}_p}) \\
 \downarrow & & \downarrow \\
 H_f^1(\pi_1^{\text{ét}}(Z \setminus \{p\}), \pi_1^{[n]}(X, x)_{\mathbb{Q}_p}^{\text{ét}}) & \longrightarrow & F^0 \backslash \pi_1^{[n]}(X, x)_{\mathbb{Q}_p}^{\text{dR}}
 \end{array}$$

factoring Kim's cutter. The lower square consists of maps of presheaves.

Remark:

Given $C \in \mathcal{C}_{\text{ét}}$, a suitable étale realization
define the tower on the left...

$$\begin{array}{c}
 C \\
 \uparrow \\
 \vdots \\
 \uparrow \\
 \mathfrak{M}_2 C \longleftarrow \text{Sym } \tau^{\leq 2} V_2 \\
 \uparrow \\
 \mathfrak{M}_1 C \longleftarrow \text{Sym } \tau^{\leq 2} V_1 \\
 \uparrow \\
 \mathbb{1} \longleftarrow \text{Sym } \tau^{\leq 2} V_0
 \end{array}$$

$$V_n^{\mathfrak{M}} \longrightarrow \mathfrak{M}_n C \longrightarrow C \longrightarrow V_n^{\mathfrak{M}}[1]$$

$$\begin{array}{ccc}
 \mathfrak{M}_{n+1} C & \longleftarrow & \mathbb{1} \\
 \uparrow & & \uparrow \\
 \mathfrak{M}_n C & \longleftarrow & \text{Sym } \tau^{\leq 2} V_n
 \end{array}$$

by the exact triangles in $\mathcal{D}_{\text{ét}}$ and the pushout squares in $\mathcal{C}_{\text{ét}}$ on the right.

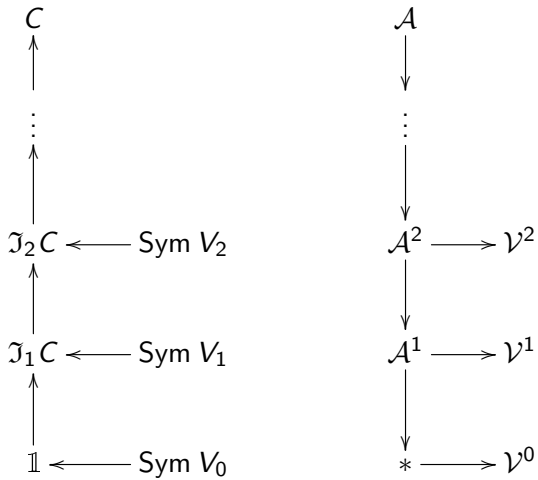
Then

- $\mathfrak{M}_i C$ maps to $\mathfrak{I}_i C^{\text{ét}}$
- $\mathfrak{M}_i C$ realizes to the classical *1-minimal* filtration, which lifts the dcs.

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Consider the Iwanari tower and the dual tower of presheaves of augmentations ($\mathcal{A}^n = \mathcal{A}ug^n(C)$):



Theorem C

Assume

- V_i is a sum of compact objects for all $0 \leq i \leq n-1$,
- $\pi_2 \mathcal{V}^i(\mathbb{Q}) = 0$ for all $0 \leq i \leq n-2$, and
- $\pi_1 \mathcal{V}^i(\mathbb{Q}), \pi_0 \mathcal{V}^i(\mathbb{Q})$ are finite dimensional for all $0 \leq i \leq n-1$.

Then

- 1 $\pi_0 \mathcal{A}^n$ is representable by a finite-type affine $\mathbb{Q}$ -scheme,
- 2 for every $\mathbb{Q}$ -algebra R , every connected component of $\mathcal{A}^{n-1}(R)$ is simply connected, and
- 3 we have

$$\dim \pi_0 \mathcal{A}^n \leq \sum_{i=0}^{n-1} \dim \pi_1 \mathcal{V}^i(\mathbb{Q}),$$

with equality if $\pi_0 \mathcal{V}^i(\mathbb{Q}) = 0$ for $i = 1, \dots, n-1$.

Example (sketch): $X = E \setminus \{0\}$ a punctured elliptic curve over $Z \subset \operatorname{Spec} \mathbb{Z}$. Iwanari computes $\mathfrak{I}_3 C = C^*(X)$ as follows. The portion $\mathfrak{I}_{\leq 2}$ of the tower (solid arrows):

$$\begin{array}{ccc}
 C = \mathbb{Q}(0) \oplus h^1(E)[-1] & & \\
 \uparrow & & \\
 \vdots & & \\
 \uparrow & & \\
 \mathfrak{I}_2 C & \xleftarrow{\cdots} & \mathbb{1} \\
 \uparrow & & \uparrow \\
 \mathbb{Q}(0) \oplus h^1(E)[-1] \oplus \mathbb{Q}(-1)[-2] = \operatorname{Sym} h^1(E)[-1] & \xleftarrow{\cdots} & \operatorname{Sym} \mathbb{Q}(-1)[-2] \\
 \uparrow & & \\
 \mathbb{1} & &
 \end{array}$$

with $\mathfrak{I}_2 C$ given by a pushout (dotted arrows).

We have, $\wedge^3 h^1(E) = 0$. It follows from Tannakian considerations obtained for instance by Iwanari, that there's a symmetric monoidal colimit preserving functor

$$\mathcal{D} \operatorname{Rep}_{\mathbb{Q}} \operatorname{GL}_2 \rightarrow \mathcal{D}_M$$

Sending

$$V_{\text{standard}} \mapsto h^1(E).$$

Hence, the pushout is given by a GL_2 -equivariant Hirsch extension. We find that

$$\mathfrak{I}_2 C \simeq \mathbb{Q}(0) \oplus h^1[-1] \oplus V_2$$

where

$$V_2 = H^1(-1)[-2] \oplus \mathbb{Q}(-2)[-3].$$

Gathering what we know about the corresponding K-groups, we find that the lower portion of the tower and dual tower are given by

$$\begin{array}{ccc}
 & & \mathcal{A}^3 \\
 & & \downarrow \\
 \mathfrak{I}_3 C & & \mathcal{A}^2 \longrightarrow K_{\bullet+1}^{(2)}(X) \\
 \uparrow & \longleftarrow \text{Sym}(H^1(-1)[-2] \oplus \mathbb{Q}(-2)[-3]) & \downarrow \\
 \mathfrak{I}_2 C & & E(Z)_{\mathbb{Q}} \longrightarrow K(\mathcal{O}_{Z,\mathbb{Q}}^*, 1) \\
 \uparrow & \longleftarrow \text{Sym}(h^1(E)[-1]) \longleftarrow \text{Sym}(\mathbb{Q}(-1)[-2]) & \downarrow \\
 \text{Sym}(h^1(E)[-1]) & & * \longrightarrow K(E(Z)_{\mathbb{Q}}, 1) \\
 \uparrow & \longleftarrow \text{Sym}(h^1(E)[-2]) & \\
 \mathbb{1} & &
 \end{array}$$

The most relevant homotopy groups are as follows:

$$\begin{array}{ccccc}
 * & \longrightarrow & K_2^{(2)}(X) & \circlearrowleft & \pi_0 \mathcal{A}^3 \\
 & & & & \downarrow \\
 * & \longrightarrow & \mathcal{O}_{Z, \mathbb{Q}}^* & \circlearrowleft & \pi_0 \mathcal{A}_2 \longrightarrow * \\
 & & & & \downarrow \\
 & & & & E(Z)_{\mathbb{Q}} \longrightarrow * \\
 & & & & \downarrow \\
 & & & & *
 \end{array}$$

We obtain a motivic, potentially effective, proof of the following special case of Siegel's theorem. Let $s = \sharp(\mathrm{Spec} \mathbb{Z} \setminus Z)$. Let $r = \mathrm{rk} E(Z)$. If

$$s + r + k_2^{(2)}(X) < 4,$$

then $X(Z)$ is finite.

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Assume $\pi_0 \mathcal{A}^n$, $\pi_0 \mathcal{V}^n$ are representable. Let

$$L_n \subset \pi_0 \mathcal{A}^n \rightarrow \pi_0 \mathcal{V}^n$$

be the fiber above 0. Then $\mathcal{A}^n|_{L_n}$ admits an L_n -valued basepoint x over the identity map of L_n , and there's an induced map of presheaves of homotopy groups over L_n (solid arrows)

$$\begin{array}{ccccc} \pi_1(\mathcal{A}^n|_{L_n}, x) & \longrightarrow & \pi_1(L_n \times \mathcal{V}^n, 0) & \cdots \longrightarrow & Q_n \\ & \searrow & \downarrow & \swarrow \cdots & \\ & & L_n & & \end{array}$$

Let Q_n be the cokernel.

Proposition D

The map $\pi_0 \mathcal{A}^{n+1} \rightarrow L_n$ is a trivial Q_n -torsor.

So if Q_n is representable, so is $\pi_0 \mathcal{A}^{n+1}$.

Example (sketch): $Z \subset \operatorname{Spec} \mathcal{O}_K$.

$X = \mathbb{A}^2 \setminus \{x = 0, y = 0, \text{ or } x + y = 1\}$. Let $\mathbb{E}_i = \operatorname{Ext}_Z^1(\mathbb{Q}(0), \mathbb{Q}(i))$.

The lower portion of the Iwanari tower and the dual tower take the following form:

$$\mathbb{Q}(0) \oplus \mathbb{Q}(-1)[-1]^{\oplus 3} \oplus \mathbb{Q}(-2)[-2]^{\oplus 3}$$



$$\mathfrak{I}_2 C$$



$$\operatorname{Sym}(\mathbb{Q}(-1)[-1]^{\oplus 3} \oplus \mathbb{Q}(-2)[-2]^{\oplus 3}) \ll \operatorname{Sym}(\mathbb{Q}(-2)[-2]^{\oplus 3} \oplus V')$$



$$\mathcal{A}^2$$



$$K(\mathbb{E}_1, 0)^3 \times K(\mathbb{E}_2, 1)^3 \twoheadrightarrow K(\mathbb{E}_2, 1)^3 \times \mathcal{V}'$$

(*)

with V' a sum of shifts of Tate motives $\mathbb{Q}(-i)[-i]$ with $i \geq 2$, so that $\mathcal{V}'$ is connected and simply connected.

We find that Q_1 is the cokernel of a map of trivial vector groups of rank 3:

$$\begin{array}{c} \mathbb{E}_1^3 \times \mathbb{E}_2^3 \xrightarrow{\psi} \mathbb{E}_1^3 \times \mathbb{E}_2^3 \longrightarrow Q_1 \quad \circlearrowright \quad \pi_0 \mathcal{A}^2 \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \downarrow \phi \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \mathbb{E}_1^3. \end{array}$$

Proposition

The map ψ is an isomorphism, so $Q_1 = 0$ and ϕ is an isomorphism.

We've found that $\pi_0 \mathcal{A}ug^2(X)$ is representable, albeit not very interesting.

Steps towards an iterative *method*. Assume the unit $\mathbb{1} \rightarrow C$ splits in $\mathcal{D}_M$. The exact couple of homotopy presheaves (e.g. over L_3) may be arranged into a grid of knights moves:

$$\begin{array}{ccccccc}
 \pi_3 \mathcal{A}^1 & \xrightarrow{\alpha_3} & \pi_3 \mathcal{V}^1 & \rightsquigarrow & \pi_2 \mathcal{A}^2 & \xrightarrow{\alpha_2} & \pi_2 \mathcal{V}^2 \rightarrow \pi_1 \mathcal{A}^3 \xrightarrow{\alpha_1} \pi_1 \mathcal{V}^3 \rightarrow \pi_0 \mathcal{A}^4 \rightarrow \pi_0 \mathcal{V}^4 \\
 & & \downarrow \text{wavy} & & \downarrow & & \downarrow \\
 & & \pi_2 \mathcal{A}^1 & \rightsquigarrow & \pi_2 \mathcal{V}^1 & \rightsquigarrow & \pi_1 \mathcal{A}^2 \rightarrow \pi_1 \mathcal{V}^2 \rightarrow \pi_0 \mathcal{A}^3 \rightarrow \pi_0 \mathcal{V}^3 \\
 & & & & \downarrow \text{wavy} & & \downarrow \\
 & & & & \pi_1 \mathcal{A}^1 & \rightsquigarrow & \pi_1 \mathcal{V}^1 \rightsquigarrow \pi_0 \mathcal{A}^2 \rightarrow \pi_0 \mathcal{V}^2 \\
 & & & & & & \downarrow \text{wavy} \\
 & & & & & & \pi_0 \mathcal{A}^1 \rightsquigarrow \pi_0 \mathcal{V}^1 \\
 & & & & & & \downarrow \\
 & & & & & & *
 \end{array}$$

“Unipotent” will mean *representable by a unipotent group scheme over L_3* . (Most of them will be vector groups.)

- We assume the homotopy groups $\pi_{\geq 1}$ below the top row have already been shown to be unipotent.
- We assume $\pi_0 \mathcal{A}_3$ is representable.

We then have a chain of weak implications:

$\pi_3 \mathcal{A}^1$ unipotent (so α_3 is a map of unipotent groups)



$\text{cok } \alpha_3$ is unipotent



$\pi_2 \mathcal{A}^2$ is unipotent (α_2 is a map of unipotent groups)



$\text{cok } \alpha_2$ is unipotent



$\pi_1 \mathcal{A}^3$ is unipotent (α_1 is a map of unipotent groups)



$\text{cok } \alpha_1 = Q_3$ is unipotent



$\pi_0 \mathcal{A}^4$ is representable.

Appendices

- viii. The motivic cochain algebra of an abelian variety
- ix. The motivic cochain algebra of a $\mathbb{G}_m$ -torsor: statement
- x. The motivic cochain algebra of a $\mathbb{G}_m$ -torsor: proof

The de Rham dga of an abelian variety $A_{\mathbb{C}}$ over $\mathbb{C}$ is equivalent to the cdga with zero differential

$$C_{\mathrm{dR}}^*(A_{\mathbb{C}}) \simeq \mathrm{Sym} H_{\mathrm{dR}}^1(A_{\mathbb{C}}).$$

A similar statement holds motivically:

- Ancona et. al. construct $M_1(A) \in \mathcal{D}_M$ by considering the sheaf of $\mathbb{Q}$ -vector spaces $A \otimes \mathbb{Q}$.
- Let $M^1 = M_1^\vee$.
- Iwanari shows that

$$C(A) = \mathrm{Sym} M^1(A).$$

Thus,

$$\mathcal{A}ug(A) = \mathrm{Hom}_{\mathcal{D}_M}(M^1(A), \mathbb{Q}(0)) = \mathrm{Hom}_{\mathcal{D}_M}(\mathbb{Q}(0), M_1(A)) = A(K) \otimes \mathbb{Q}.$$

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If $Y_{\mathbb{C}}$ is smooth over $\mathbb{C}$ and $L_{\mathbb{C}} \rightarrow Y_{\mathbb{C}}$ is a line bundle with first Chern class $c_{\mathrm{dR}} \in H_{\mathrm{dR}}^2(Y_{\mathbb{C}})$ and associated $\mathbb{G}_m$ -torsor L^* , then

$$\Omega_{C^\infty}^\bullet(L^*) \simeq \Omega_{C^\infty}^\bullet(Y)[t]/(dt - c_{\mathrm{dR}}).$$

More precisely, if

$$\mathbb{C}[-2] \xrightarrow{c_{\mathrm{dR}}} C_{\mathrm{dR}}^*(Y_{\mathbb{C}}) \rightarrow E$$

is the cofiber of c_{dR} then the induced square

$$\begin{array}{ccc} \mathrm{Sym} C_{\mathrm{dR}}^*(Y_{\mathbb{C}}) & \longrightarrow & \mathrm{Sym} E \\ \downarrow & & \downarrow \\ C_{\mathrm{dR}}^*(Y_{\mathbb{C}}) & \longrightarrow & C_{\mathrm{dR}}^*(L_{\mathbb{C}}^*) \end{array}$$

is a homotopy pushout square.

Now let Y be smooth over Z and L a line bundle over Y . Let

$$c^L : M^1(\mathbb{G}_m)[-1] = \mathbb{Q}(-1)[-2] \rightarrow C^*(Y)$$

be the motivic first Chern class, and let

$$\mathbb{Q}(-1)[-2] \rightarrow C^*(Y) \rightarrow E \rightarrow \mathbb{Q}(-1)[-1] = M^1(\mathbb{G}_m)$$

be the cofiber in $\mathcal{D}$.

Theorem E (Joint with L. A. Betts)

The square

$$\begin{array}{ccc} \mathrm{Sym} C^*(Y) & \longrightarrow & \mathrm{Sym} E \\ \downarrow & & \downarrow \\ C^*(Y) & \longrightarrow & C^*(L^*) \end{array}$$

is (homotopy) coCartesian.

- Hence, the augmentation space sits in a homotopy pullback square

$$\begin{array}{ccc}
 \mathrm{Hom}_{\mathcal{D}}(C^*(Y), \mathbb{Q}(0)) & \leftarrow & \mathrm{Hom}_{\mathcal{D}}(E, \mathbb{Q}(0)) \leftarrow \mathcal{A}ug(\mathbb{G}_m) = \mathcal{O}_Z^* \otimes \mathbb{Q} \\
 \uparrow & & \uparrow \\
 \mathcal{A}ug(Y) & \longleftarrow & \mathcal{A}ug(L^*)
 \end{array}$$

- It follows that $\mathcal{A}ug(L^*)$ has the structure of an $\mathcal{O}_Z^* \otimes \mathbb{Q}$ -torsor over $\mathcal{A}ug(Y)$.

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Let P be the pushout

$$\begin{array}{ccc} \mathrm{Sym} C^*(Y) & \longrightarrow & \mathrm{Sym} E \\ \downarrow & & \downarrow \\ C^*(Y) & \longrightarrow & P. \end{array}$$

Then by base-changing everything to $C^*(Y)$ we find that P is also a pushout

$$\begin{array}{ccc} \mathrm{Sym}_{C^*(Y)} C^*(Y) & \longrightarrow & \mathrm{Sym}_{C^*(Y)} C^*(L^*) \\ \downarrow & & \downarrow \\ C^*(Y) & \longrightarrow & P. \end{array}$$

$$\begin{array}{ccc}
 \mathrm{Sym}_{C(Y)} C(Y) & \longrightarrow & \mathrm{Sym}_{C(Y)} C(L^*) \\
 \downarrow & & \downarrow \\
 C(Y) & \longrightarrow & P.
 \end{array}$$

The copy of $C(L^*)$ in the upper right comes from the Gysin sequence

$$C(Y)(-1)[-2] \xrightarrow{\text{mult. by } c^L} C(Y) \rightarrow C(L^*) \rightarrow C(Y)(-1)[-1]. \quad (*)$$

This shows that P may be identified with the “relatively free commutative algebra” generated by the pointed object

$$C(Y) \rightarrow C(L^*)$$

of $\mathrm{Mod}_{C(Y)}$.

Lurie's theory of free algebras shows that

$$P = \operatorname{colim} C(L^*)^{c^L \otimes}$$

is given by the colimit of a diagram

$$C(L^*)^{c^L \otimes} : \operatorname{Fin}_{\operatorname{inj}} \rightarrow \operatorname{Mod}_{C(Y)}$$

which mixes the symmetric group actions on the tensor powers of $C(L^*)$ with multiplication by the 1st Chern class c^L . Using the fact that $\operatorname{Mod}_{C(Y)}$ is tensored over rational spaces, we may decompose the colimit as

$$P = \operatorname{colim}_n \operatorname{Sym}_{C(Y)}^n C(L^*).$$

Taking symmetric powers over $C(Y)$ in the exact triangle

$$C(Y)(-1)[-2] \xrightarrow{\text{mult. by } c^L} C(Y) \rightarrow C(L^*) \rightarrow C(Y)(-1)[-1], \quad (*)$$

we get exact triangles

$$\mathrm{Sym}_{C(Y)}^{n-1} C(L^*) \rightarrow \mathrm{Sym}_{C(Y)}^n C(L^*) \rightarrow \mathrm{Sym}_{C(Y)}^n C(Y)(-1)[-1].$$

We have

$$\mathrm{Sym}_{C(Y)}^n C(Y)(-1)[-1] = 0 \quad \text{for } n \geq 2.$$

So finally,

$$P = \mathrm{colim}(C(Y) \rightarrow C(L^*) \xrightarrow{=} C(L^*) \xrightarrow{=} C(L^*) \xrightarrow{=} \cdots) = C(L^*).$$