

# Towards $p$ -adic periods in Chabauty-Kim theory

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Joint with D. Corwin.

- ▶ We'll focus on the case of the complement of the zero section inside an elliptic scheme over an open subscheme of  $\text{Spec } \mathbb{Z}$ :

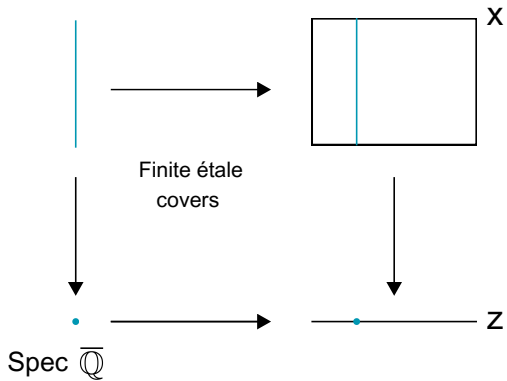
$$X \subset E \rightarrow Z \subset \text{Spec } \mathbb{Z}.$$

- ▶ Grothendieck's theory of the étale fundamental group suggests one way to think about the sections of

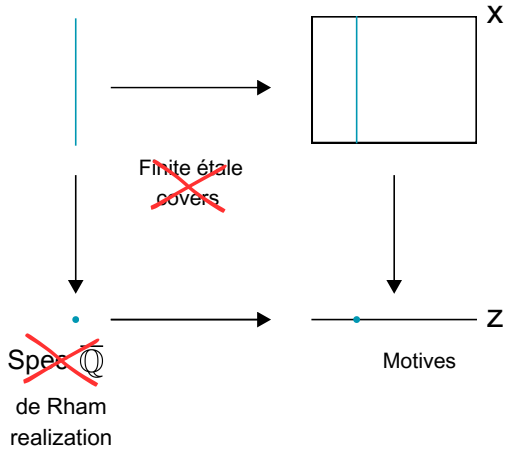
$$X \rightarrow Z$$

("integral points").

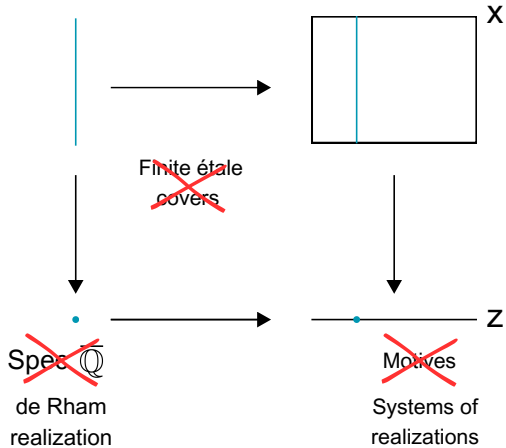
- ▶ Kim's approach to the study of integral points suggests a different way.



Vector bundles with  
integrable connection on  $X_{\mathbb{Q}}$



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# Systems of realizations

- ▶ There are many realizations of  $H_1(E)$ : a mixed Hodge structure, Galois representations, a filtered  $\phi$  module; these are related by various comparison isomorphisms.
- ▶ Better:

$$H_1(E) \in \text{MMS}(Z)$$

belongs to the Tannakian category of *mixed motivic systems of realizations over  $Z$  with  $\mathbb{Q}$  coefficients*.

# Mixed elliptic systems and the unipotent fundamental group

- ▶ Let

$$M_{\mathbb{E}}S(Z) \subset MMS(Z)$$

be the Tannakian subcategory generated by  $H_1(E)$ , closed under extensions.

- ▶ We fix a point  $b$  for use as base-point.
- ▶ The *unipotent fundamental group*  $\tilde{U}(X, b)$  is a prounipotent group object in  $M_{\mathbb{E}}S(Z)$ . We'll actually work with a unipotent quotient  $U(X, b)$ .

# Properties

- ▶ The Betti realization  $\tilde{U}(X, b)_B$  is the prounipotent completion of the topological fundamental group.
- ▶ The de Rham realization  $\tilde{U}(X, b)_{\text{dR}}$  is the Tannakian fundamental group of the category of unipotent vector bundles with integrable connection on  $X_{\mathbb{Q}}$ .
- ▶ Fix a prime  $p \in Z$ . The *filtered  $\phi$  realization*  $U(X, b)_{F\phi}$  is the de Rham realization over  $\mathbb{Q}_p$  equipped with some naturally occurring extra structures: a “Hodge” filtration and a “Frobenius” automorphism.
- ▶ A point  $x \in X(Z)$  gives rise to a torsor object  $U(X, b, x)$ .

- ▶ To motivate our constructions, recall that the unipotent  $p$ -adic Albanese map

$$\alpha : X(\mathbb{Z}_p) \rightarrow F^0 \backslash U(X, b)_{F\phi}$$

is defined in terms of two special sorts of de Rham paths.

- ▶ We'll refer to these as *geometric* paths, in contrast to the *arithmetic* paths we'll introduce later.

## Special geometric paths

- ▶ The  $p$ -adic de Rham realization of the path torsor  $U(X, b, x)_{F\phi}$  is defined more generally for  $x \in X(\mathbb{Z}_p)$ .
- ▶ The subspace

$$F^0 U(X, b, x)_{F\phi} \subset U(X, b, x)_{F\phi}$$

forms a torsor under  $F^0 U(X, b)_{F\phi}$

- ▶ (these are the “geometric Hodge paths”).
- ▶ There’s a unique Frob. fixed path  $p^{cr} \in U(X, b, x)_{F\phi}$
- ▶ (the “geometric crystalline path”).
- ▶ The *unipotent  $p$ -adic Albanese map*

$$\alpha : X(\mathbb{Z}_p) \rightarrow F^0 \backslash U(X, b)_{F\phi}$$

is given by

$$\alpha(x) = (p^H)^{-1} \circ p^{cr}$$

for any geometric Hodge path  $p^H \in F^0 U(X, x, b)_{F\phi}$ .

## Relationship to iterated integrals

- ▶ An element  $\omega \in T^\bullet H^1(E)_{\text{dR}}$  of the tensor algebra may be used to define a function

$$f_\omega : U(X, b)_{\text{dR}} \rightarrow \mathbb{A}_{\mathbb{Q}}^1;$$

- ▶ if  $f_\omega$  factors through  $F^0 \setminus U(X, b)$ , then the composition

$$X(\mathbb{Z}_p) \xrightarrow{\alpha} F^0 \setminus U(X, b)_{F\phi} \xrightarrow{f_\omega} \mathbb{A}_{\mathbb{Q}_p}^1$$

is given by the  $p$ -adic iterated integral

$$x \mapsto \int_b^x \omega.$$

## Structure of arithmetic Tannakian Galois group

- ▶ The Tannakian Galois group  $G(M_{\text{ES}}, dR) = G(dR) = G(Z)$  sits in a noncanonically split short exact sequence

$$1 \rightarrow U(Z) \rightarrow G(Z) \rightarrow \mathbb{G} \rightarrow 1$$

with  $\mathbb{G}$  reductive and  $U(Z)$  prounipotent,

- ▶ the “unipotent fundamental group of  $Z$ ”.
- ▶ The sequence becomes canonically split if we replace  $dR$  by  $dR \circ \text{gr}^W$ .
- ▶ If  $E$  does *not* have CM, then  $\mathbb{G} = GL_2$ .
- ▶ We let

$$F^0 U(dR) \subset U(dR)$$

be the subgroup of Tannakian loops which preserve Hodge filtrations.

## Axiomatic setting for Propositions A, B, C

- ▶ The category  $M_{\text{ES}}(Z)$  is an example of a *weight filtered Tannakian category*  $T$ . (Objects are equipped with filtrations, morphisms are strict, pure objects are semisimple.)
- ▶ De Rham realization is in a natural way a *Hodge filtered fiber functor*  $\omega^H : T \rightarrow \text{FilVect}(\mathbb{Q})$ . In this axiomatic setting, let's denote the associated fiber functor (de Rham realization) by  $\omega$ .
- ▶ Crystalline realization is in a natural way a *Frobenius equivariant fiber functor*

$$\omega^{cr} : T \rightarrow \phi \text{Vect}(\mathbb{Q}_p).$$

(The operator  $\phi$  is required to act with eigenvalues of weight  $n$  on  $\text{gr}_n^W M$ .)

## Proposition (A)

- ▶ The space of  $\otimes$ -compatible natural transformations

$$\begin{array}{ccc} & \omega^H \circ \text{gr}^W & \\ & \curvearrowright & \\ T & & \text{FilVect}(\mathbb{Q}) \\ & \curvearrowleft & \\ & \omega^H & \end{array} \quad \begin{array}{c} \Downarrow p^H \end{array}$$

which induce the identity on associated graded objects, forms a trivial torsor under  $F^0 U(\omega)$ .

- ▶ These are our “arithmetic Hodge paths”.

## Proposition (B)

- ▶ There exists a unique  $\otimes$ -compatible natural transformation

$$\begin{array}{ccc} & \xrightarrow{\omega^{cr} \circ \text{gr}^W} & \\ T & \searrow & \phi \text{Vect}(\mathbb{Q}_p) \\ & \xrightarrow{\omega^{cr}} & \end{array} \quad \Downarrow p^{cr}$$

which induces the identity on associated graded objects.

- ▶ This is our “arithmetic crystalline path”.

## Construction of $p$ -adic period loop

- ▶ We fix an arithmetic Hodge path  $\mathfrak{p}^H$ .
- ▶ :(
- ▶ We define the *unipotent  $p$ -adic period loop* by

$$u := \mathfrak{p}^{cr} \circ (\mathfrak{p}^H)^{-1}.$$

- ▶ Existence and uniqueness of geometric crystalline paths applies in our axiomatic setting to torsor objects under effective prounipotent group objects in  $T$ .

## Proposition C

- ▶ Let  $\pi$  be an effective unipotent group object in  $T$  (concentrated in negative weights)
- ▶ and let  $P$  be a torsor object with Frobenius fixed point  $p^{cr}$ .
- ▶ Then

$$u^{-1}p^{cr} \in F^0P_\omega.$$

- ▶ Said differently, our unipotent  $p$ -adic period loops interchange geometric crystalline paths and geometric Hodge paths.
- ▶ Thus, the choice of arithmetic Hodge path gives us consistent choices of geometric Hodge paths.
- ▶ Wherever we encounter a geometric crystalline path  $p^{cr}$ , we set

$$p^H := u^{-1} p^{cr}.$$

- ▶ Our main application takes place in an intermediate level of generality.
- ▶ For instance,  $X$  may be a suitable model of a hyperbolic curve with motivic Mumford-Tate group  $\mathbb{G}$  (fundamental group of the category of pure motivic systems generated by  $H_1$  of a compactification).

- ▶ We obtain the following

### Theorem (D)

We have a commutative diagram like so:

$$\begin{array}{ccc}
 X(Z) & \longrightarrow & X(\mathbb{Z}_p) \\
 \kappa \downarrow & & \downarrow \alpha \\
 Z^1(U(Z, dR), U(X)_{dR})^G & \xrightarrow{ev_u} & F^0 \setminus U(X)_{F\phi}.
 \end{array}$$

- ▶ For  $x \in X(Z)$  and  $\gamma \in U(Z, dR)$ ,

$$\kappa_x(\gamma) = (p^H)^{-1} \circ \gamma(p^H).$$

- ▶ The map  $ev_u$  is given by evaluation at  $u$ .

- ▶ We also have a factorization of the evaluation map

$$\begin{array}{ccc}
 Z^1(U(Z, \mathrm{dR}), U(X)_{\mathrm{dR}})^{\mathbb{G}} & \xrightarrow{ev_u} & F^0 \backslash U(X)_{F\phi} \\
 \downarrow & \nearrow & \\
 Z^1(U(\mathrm{GalRep}), U(X)_{p\text{-ét}})^{\mathbb{G}} & & 
 \end{array}$$

$ev_{u, p\text{-ét}}$

associated to realization into a suitable category of  $p$ -adic Galois representations.

- ▶  $Z^1(U(\mathrm{GalRep}), U(X)_{p\text{-ét}})^{\mathbb{G}}$  is the same as Kim's Selmer variety, so in particular representable by a finite type affine  $\mathbb{Q}_p$ -scheme.
- ▶  $Z^1(U(Z, \mathrm{dR}), U(X)_{\mathrm{dR}})^{\mathbb{G}}$  is conjecturally representable by a finite type affine  $\mathbb{Q}$ -variety. (But if we want to really compute the image of  $ev_u$  then representability might not play a central role anyway.)

## Corollary

We have

$$\text{Kim functions} = \alpha^\# \ker \text{ev}_{\mathbb{U}_{p\text{-ét}}}^\#.$$

## Motivic Kim functions

- ▶ It's natural to regard  $\epsilon v_u$  as the fiber at  $u$  of the family of evaluation maps at all points.
- ▶ Let  $\epsilon v =$  evaluation followed by projection

$$U(Z) \times Z^1(U(Z), U(X))^{\mathbb{G}} \rightarrow U(Z) \times F^0 \setminus U(X)^{\mathrm{dR}}.$$

- ▶ If  $f : U(Z) \times F^0 \setminus U(X)^{\mathrm{dR}} \rightarrow \mathbb{A}_{\mathbb{Q}}^1$  obeys  $f \circ \epsilon v = 0$ , then the pullback of  $f$  to  $X(\mathbb{Z}_p)$  is in an evident sense a linear combination of *rationally defined* Besser-Coleman functions whose coefficients are periods.
- ▶ One may hope to identify the coefficients of  $f$  as motivic iterated integrals.

We'll give two examples of concrete applications.

## Example (E)

- ▶ Joint work with D. Corwin and M. Lüdtkke. Let  $Z = \text{Spec } \mathbb{Z}[1/2]$ ,  $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ .
- ▶ Then  $X(\mathbb{Z}_p)$  admits the motivic Kim function

$$\begin{aligned} \log(2)\zeta(3) \text{Li}_{2,1,1}(z) &- \left( \zeta(3) \log(2) - \frac{8}{7} \text{Li}_4(2) \right) \text{Li}_1(z) \text{Li}_{2,1}(z) \\ &- \frac{1}{24} \left( \log(2)\zeta(3) - 4\frac{8}{7} \text{Li}_4(2) \right) \log(z) \text{Li}_1(z)^3 \\ &+ \zeta(3) \left( \zeta(3) \log(2) - \frac{8}{7} \text{Li}_4(2) \right) \text{Li}_1(z). \end{aligned}$$

- ▶ This function occurs in weight 8, and hence beyond the so-called *quadratic quotient*.
- ▶ In a different direction, it goes beyond the so-called *polylogarithmic quotient*.

## Period coefficients

- ▶ The word in 1-forms

$$\omega = \frac{dt}{1-t} \cdot \frac{dt}{t} \cdot \frac{dt}{t}$$

may be used to define a function

$$f_\omega : U(X, 0, 1) \rightarrow \mathbb{A}_{\mathbb{Q}}^1$$

(tangential base points).

- ▶ Precomposing with the orbit map of a geometric Hodge path (unique in this case), we obtain a function

$$\zeta^f(3) : U(\text{MTM}, d\mathbb{R}) \rightarrow \mathbb{A}^1$$

(a “formal period”)

- ▶ such that

$$\zeta^f(3)(\mathbf{u}) = \zeta(3).$$

## Remarks

- ▶ This is an example of a  $p$ -adic period of a motivic iterated integral.
- ▶ Goncharov's "universality conjecture" implies that all coefficients of motivic Kim functions in the mixed Tate setting are of this form.

## Example (F)

- ▶ Work of D. Corwin, contributions by O. Patashnik and M. Lüdtkke.
- ▶ Suppose in the example

$$X \subset E \rightarrow Z \subset \text{Spec } \mathbb{Z}$$

we have 1 inverted prime, and  $E$  has  $p$ -Selmer rank 1 and no potential CM.

- ▶ Let  $\alpha, \beta$  be a basis of  $H_{\text{dR}}^1(E_{\mathbb{Q}})$  with  $\alpha$  holomorphic and  $\beta$  of type II.
- ▶ Then there exists a nonzero Kim function of the form

$$\begin{aligned} c_1 \int_0^x (\alpha\beta\beta - 2\beta) + c_2 \int_0^x \alpha\beta\alpha + c_3 \int_0^x (2\alpha\alpha\beta + \alpha\beta\alpha) \\ + c_4 \int_0^x \alpha\alpha\alpha + c_5 \int_0^x \alpha. \end{aligned}$$

## Proof of (F)

- ▶ We have  $U(Z, dR)^{\text{ab}} = \prod_{i,j} \text{Ext}^1(\mathbb{Q}(0), \text{Sym}^i H_1(E)(j))^\vee \otimes \text{Sym}^i H_1^{\text{dR}}(E)(j)$ .
- ▶ There's a motivic quotient  $U(X)$  of the unipotent fundamental group such that

$$\text{Lie } U(X) = H_1(E) \oplus \mathbb{Q}(1) \oplus H_1(E)(1) =: W.$$

- ▶ In the category  $M_E \text{GRep}(Z)$  of mixed elliptic  $p$ -adic Galois representations which are unramified over  $Z \setminus \{p\}$  and crystalline at  $p$ , we have

$$\begin{aligned} \dim \text{Ext}^1(\mathbb{Q}_p(0), H_1(E)) &= \dim \text{Ext}^1(\mathbb{Q}_p(0), \mathbb{Q}_p(1)) \\ &= \dim \text{Ext}^1(\mathbb{Q}_p(0), H_1(E)(1)) = 1. \end{aligned}$$

- ▶ These calculations allow us to replace the unipotent Tannakian Galois group by the free prounipotent group with  $\text{GL}_{2, \mathbb{Q}_p}$ -action  $U(W_{p\text{-ét}})$ .

## Proof of (F)

We also regard  $\mathfrak{ev}_u$  as the fiber at  $u$  of the family of all evaluation maps:

$$\begin{array}{ccc} X(Z) & \longrightarrow & X(\mathbb{Z}_p) \\ \kappa \downarrow & & \downarrow \alpha \\ Z^1(U(Z), U(X)_{p\text{-ét}})^{\mathbb{G}} & \xrightarrow{\mathfrak{ev}_u} & F^0 \backslash U(X)_{F\phi} \\ \downarrow & & \downarrow \\ U(Z) \times Z^1(U(Z), U(X)_{p\text{-ét}})^{\mathbb{G}} & \xrightarrow{\mathfrak{ev}} & U(Z) \times F^0 \backslash U(X)_{F\phi} \end{array}$$

## Proof of (F)

Using the above, we eventually reduce to the following purely geometric problem.

- ▶ Let  $V$  be the standard representation of  $GL_2$  over a field  $k$ .
- ▶ The representation

$$W = V \oplus \det \oplus V \otimes \det,$$

carries a natural structure of nilpotent Lie algebra.

- ▶ Let  $U^g$  be the associated unipotent group.
- ▶ Let

$$F^0 U^g \subset U^g$$

be a subgroup given by certain explicit equations (work of J. Beacom).

- ▶ Let  $U^a$  be free pronipotent on  $W$ .

## Proof of (F)

- ▶ Finally, consider the universal evaluation map

$$\text{ev} : Z^1(U^a, U^g)^{\text{GL}_2} \times U^a \rightarrow U^g \times U^a$$

$$\text{ev}(c, u) = (c(u), u).$$

- ▶ **Problem:** Find a finite free  $\mathcal{O}(U^a)$ -submodule  $I$  of

$$\mathcal{O}(F^0 \setminus U^g) \otimes \mathcal{O}(U^a)$$

such that  $\text{ev}^\# I$  is contained in a finite free  $\mathcal{O}(U^a)$ -submodule of

$$\mathcal{O}(Z^1(U^a, U^g)^{\text{GL}_2}) \otimes \mathcal{O}(U^a)$$

of strictly lower rank.

- ▶ This sort of problem is easily solved in minimal examples (e.g. in the setting of the present example).

Proof of Proposition B (existence and uniqueness of arithmetic crystalline paths).

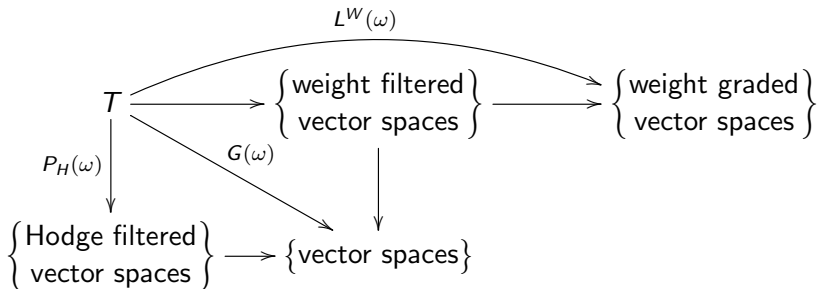
- ▶ Let  $G(\omega)$  act on  $U(\omega)$  by conjugation.
- ▶ We find that  $U(\omega)$  is concentrated in negative weights.
- ▶ By Besser's argument, the Lang map

$$\begin{aligned} U(\omega) &\rightarrow U(\omega) \\ g &\mapsto g^{-1}\phi(g) \end{aligned}$$

is iso.

- ▶ This implies that every  $G(\omega)$ -equivariant  $U(\omega)$ -torsor possesses a unique  $\phi$ -fixed point.
- ▶ We realize  $\mathfrak{p}^{\text{cr}}$  as the unique  $\phi$ -fixed point in a certain  $G(\omega)$ -equivariant  $U(\omega)$ -torsor.

Proof of Proposition A (existence of arithmetic Hodge paths).  
 We consider the following groups of  $\otimes$ -automorphisms of symmetric monoidal functors:



Via Tannaka duality, the problem translates into the problem of lifting an associated central cocharacter:

$$\begin{array}{ccc} & & P_H(\omega) \\ & \nearrow & \cap \\ & & G(\omega) \\ & & \downarrow \\ \mathbb{G}_m & \xrightarrow{\chi^W} & L^W(\omega) \end{array}$$

The kernel of the vertical map is unipotent. Work of P. Ziegler shows that in the finite type case, the groups on the right all have the same rank.

## Unipotent $p$ -adic periods

- ▶ Our unipotent  $p$ -adic period loops give us a well defined point

$$u : \text{Spec } \mathbb{Q}_p \rightarrow U(\text{dR})/F^0.$$

- ▶ Equivalently, a homomorphism defined on a certain subalgebra

$$\mathbb{Q}_p \xleftarrow{\text{per}} \mathcal{O}(U(\text{dR})/F^0) \subset \mathcal{O}(U(\text{dR})).$$

- ▶ We call elements in its image *unipotent  $p$ -adic periods*.

## Example

- ▶ If  $\alpha$  is a global nonvanishing 1-form on  $E_{\mathbb{Q}}$  normalized with respect to a nonzero tangent vector  $v$  at the origin,
- ▶ and  $x \in E(\mathbb{Q})$ ,
- ▶ then there's a function

$$f_{x,\alpha} : U(dR)/F^0 \rightarrow \mathbb{A}_{\mathbb{Q}}^1$$

- ▶ such that

$$f_{x,\alpha}(\mathbf{u}) = \int_v^x \alpha.$$

# Conjecture

- ▶ A natural extension of the  $p$ -adic period conjecture for mixed Tate motives says that the ring homomorphism

$$\text{per} : \mathcal{O}(U(\text{dR})/F^0) \rightarrow \mathbb{Q}_p$$

is injective.

- ▶ Compare recent work by Ancona–Frățilă.

## Evidence? (Conversations with Wendt and Moore)

- ▶  $E$  = a suitable model of an elliptic curve over a quadratic imaginary field  $K$  with CM by the full ring of integers  $\mathcal{O}_K$ .
- ▶  $p > 5$  a prime of good reduction unramified on  $\mathcal{O}_K$  with local field  $K_p$ .
- ▶  $z_0 \in E(K)$  torsion of order prime to  $p$ .
- ▶ Bannai et. al. use the “polylogarithm sheaf” of  $E$  to construct certain transcendental numbers  $e_{i,j}^{(p)}(z_0)$ .
- ▶ Expected to intervene in an analog of Zagier’s conjecture.
- ▶ Work of Kings and Solomon may be used to show that there are two, a priori different, collections of functions  $\mathfrak{e}_{i,j}(z_0)$  and  $\mathfrak{e}_{i,j}^*(z_0) \in \mathcal{O}(U(\mathrm{dR})^{\mathrm{ab}}/F^0)$  such that

$$\mathfrak{e}_{i,j}(z_0)(\mathbf{u}) = e_{i,j}^{(p)}(z_0) = \mathfrak{e}_{i,j}^*(z_0)(\mathbf{u}).$$

## Evidence? (Conversations with Wendt and Moore)

- ▶ The specialization of the polylog sheaf at  $z_0$  gives rise to extensions

$$0 \rightarrow \mathcal{L}^j \rightarrow \mathcal{P}_{z_0}^j \rightarrow \mathbb{Q}(0) \rightarrow 0 \quad (*)$$

in  $M_E S(Z, K)$ .

- ▶ The motivic system  $\mathcal{L}^j := H^1(E) \otimes (\text{Sym}^j H_1(E))(1)$  is pure, hence semisimple.
- ▶ Its isotypic components decompose canonically as a sum of two simple objects of rank 1.
- ▶ The period conjecture predicts that for each isotypic component, the two projections of the extension class (\*) agree.

## Poor persons's motivic iterated integrals

- ▶  $T \subset E(Z)$  finite.
- ▶  $a, b$ ,  $Z$ -integral base-points of  $E \setminus T$ .
- ▶ May use  $p^H$  to trivialize  ${}_b U_a(E \setminus T)^{\mathrm{dR}}$ .
- ▶ A word  $\omega$  in suitable 1-forms  $\alpha, \beta_t$  ( $t \in T$ ) gives rise to a function

$$U_{\mathrm{dR}}(Z) \xrightarrow{\mathrm{Orb}(p^H)} {}_b U_a(E \setminus T)^{\mathrm{dR}} \xrightarrow{f_\omega} \mathbb{A}_K^1. \quad (I_a^b(\omega))$$

- ▶  $I_a^b(\omega)$  is well defined if  $f_\omega$  is replaced by a linear combination that factors modulo the left action of  $F^0$  on  ${}_b U_a(E \setminus T)^{\mathrm{dR}}$ .
- ▶ Expected compatibilities imply that the motivic EK numbers  $\epsilon_{i,j}(z_0)$  are well defined motivic iterated integrals.