p-ADIC HEIGHTS AND QUADRATIC CHABAUTY, FALL 2023 LEARNING SEMINAR

ABSTRACT. We learned about the construction of p-adic height functions and their application to quadratic Chabauty due to Besser, Müller, and Srinivasan [3]. I gave two talks.

1. SYLLABUS

Talk 1: Line bundles on abelian varieties. In this talk we review classical results on line bundles on abelian varieties. If a line bundle L on an abelian variety A is symmetric ($(-1)^*L \simeq L$), then the pullbacks along the sum and difference maps $s, d : A \times A \rightrightarrows A$ satisfiy

$$s^*L \otimes d^*L \simeq (\pi_1^*L)^{\otimes 2} \otimes (\pi_2^*L)^{\otimes 2} \tag{Eqn. 19}$$

and $L^{\otimes 2} \simeq (id \times \phi_L)^*\mathcal{P}$, where $\mathcal{P}$ denotes the Poicaré bundle. On the other hand, L is antisymmetric ($(-1)^*L \simeq L^{-1}$) iff its first Chern class is homologically trivial, iff the associated polarization $\phi_L : A \to \hat{A}$ is trivial, in which case

$$s^*L \simeq \pi_1^*L \otimes \pi_2^*L. \tag{Eqn. 20}$$

These, and other important facts, are recalled in §4.1 of [3]. Main references: Lang's *Fundamentals...* [6] and Bombieri-Gubler [5].

Talk 2: review of Néron functions. Let K be a finite extension of $\mathbb{Q}_v$, X smooth proper over K, and L a line bundle on X. An *(l-adic) valuation* on L is a function $v_L : L^\times(\bar{\mathbb{Q}}_v) \to \bar{\mathbb{Q}}_l$ which satisfies the equation

$$v_L(\lambda u) = \mathrm{ord}_v(\lambda) + v_L(u)$$

for all $\lambda \in \bar{\mathbb{Q}}_v^*$ and $u \in L^\times(\bar{\mathbb{Q}}_v^*)$. There's a unique way to assign valuations to rigidified line bundles on abelian varieties so that v_L is additive in L, functorial in A, locally constant on $L^\times$, is $\mathbb{Q}$-valued with bounded denominator on $L^\times(K)$, vanishes on the base-point, and reduces to ord_v on the trivial bundle. When A has good reduction, this reduces to naive intersection in the Néron model. Confusingly, here $v = l$; later we'll use a p-adic idele class character to move the values to $\mathbb{Q}_p$. See §2 of [3]. Main reference: Bombieri-Gubler [5].

Talk 3: Vologodsky functions and Besser's $\bar{\partial}$-operator. Let K be a finite extension of $\mathbb{Q}_p$. Locally free sheaves $\mathcal{F}$ on smooth geometrically connected K-schemes X admit Zariski sheaves $\mathcal{F}_V \in \mathrm{Mod}(\mathcal{O}_X)$ of *Vologodsky sections*. There are embeddings

$$\Omega^i(X) \subset \Omega^i_V(X) \subset \Omega^i_{\text{loc.an.}}(X).$$

There are exact sequences of *K-vector spaces*

$$0 \to K \to \mathcal{O}_V(X) \to \Omega^1_V(X) \to \Omega^2_V(X) \to \dots$$

These are functorial in X and $\mathcal{F}$, and include the p-adic logarithm on $\mathbb{G}_m$. There are also certain submodules $\Omega^1_{V,1}(X) \subset \Omega^1_V(X)$ which sit in short exact sequences

$$0 \to \Omega^1(X) \to \Omega^1_{V,1}(X) \xrightarrow{\bar{\partial}} \Omega^1(X) \otimes H^1_{\mathrm{dR}}(X).$$

See §3.1 of [3]. References: for Vologodsky's Frobenius-fixed de Rham paths, see Definition 3.12 of Betts-Litt [4]. For their use in constructing the data listed above, see Besser's *p-Adic Arakelov geometry* [2] (with further references back to [1]).

Talk 4: Log functions and curvature forms. A *log function* on a line bundle $\pi : L \to X$ is a Vologodsky function $\log_L \in \mathcal{O}_V(L^\times)$ such that $d\log_L \in \Omega^1_{V,1}(L^\times)$ and

$$\log_L(\lambda u) = \log(\lambda) + \log_L(u)$$

for $\lambda \in \bar{\mathbb{Q}}_p^*$ and $U \in L^\times(\bar{\mathbb{Q}}_p)$. A *curvature form* for L is an element $\alpha \in \Omega^1(X) \otimes H^1_{\mathrm{dR}}(X)$ such that $\cup\alpha = c_1(L)$. There's a correspondence between log functions and curvature forms such that if α and $\log_L$ correspond, then

$$\pi^*\alpha = \bar{\partial} d\log_L$$

in $\Omega^1(L^\times) \otimes H^1_{\mathrm{dR}}(L^\times)$. This is made precise in Proposition 4.4 of Besser [2]. In this talk, we outline the proof of this proposition following loc. cit., and we work out the case of curves in more detail, following §3.3 of [3].

Talk 5: Canonical log functions over abelian varieties. A line bundle L over an abelian variety A over a p-adic field admits a canonical log function $\log_L$, depending only on the choice of a curvature form α for the Poincaré bundle $\mathcal{P}$. We impose a technical condition on α (it must be "purely mixed"). If L is symmetric, then $\log_L$ is characterized by the requirements that the isomorphism of Eqn. 19 respect the induced log functions and that the associated curvature form be given by $\frac{1}{2}(id \times \phi_L)^*\alpha$. Roughly speaking, the special case $L = \mathcal{P}$ gives us a canonical choice of log function $\log_{\mathcal{P}}$ on $\mathcal{P}$. If L is antisymmetric, then $c_1(L) = 0$, so there's an associated class $\hat{a} \in \hat{A}$, and $\log_L$ is induced from $\log_{\mathcal{P}}$ by restricting to the fiber $A \times \{\hat{a}\}$. For general L we obtain $\log_L$ by mixing log functions associated to symmetric and antisymmetric parts of $L^{\otimes 2}$. See §4.2-4.4 of [3].

Talk 6: From log functions back to Néron functions. Besser's log functions provide an analytic counterpart at places $v|p$ to the discrete input given by the p-adic valuations at primes $v \nmid p$. However, going in the opposite direction, it's possible to extract a "merely valuative" portion

$$v_L = \log_L^{(1)}$$

of Besser's log-functions. For this, instead of fixing a value of $\log p$ (the "brach of the logarithm"), we consider the latter as a formal variable, and then extract the part which is linear in $\log p$. In the case of abelian varieties , if we restrict attention to log functions which are in a suitable sense branch-independent and normalized with respect to the regidification, we obtain the canonical valuation. See Theorem 9.23 of [3], as well as Zarhin [7] for portions of the proof.

Talk 7: Global p-adic heights. Let K be a number field. We fix a well-behaved p-adic idèle class character $\chi = \sum \chi_v$; this means, among other things, that we can extract a homomorphism

$$t_{\mathfrak{p}} : K_{\mathfrak{p}} \to \mathbb{Q}_p$$

at primes $\mathfrak{p}|p$. Let X/K be smooth, projective, geometrically integral. A *p-adic adelic metric* on a line bundle L on X consists of the choice of valuations $v_{L,\mathfrak{q}}$ at primes $\mathfrak{q} \nmid p$ and log functions $\log_{L,\mathfrak{p}}$ at primes $\mathfrak{p}|p$. We require that there exist suitable models of X and L so that for all but finitely many $\mathfrak{q} \nmid p$, the valuation $v_{L,\mathfrak{p}}$ be the associated model valuation. The associated *p-adic height function* $h : X(K) \to \mathbb{Q}_p$ is given by

$$h(x) = \sum_{\mathfrak{p}|p} t_{\mathfrak{p}}(\log_{L,\mathfrak{p}}(u)) + \sum_{\mathfrak{q}\nmid p} v_{L,\mathfrak{q}}(u)\chi_{\mathfrak{q}}(\pi_{\mathfrak{q}}).$$

independently of choice of $u \in L_x^\times(K)$.

A line bundle over an abelian variety admits a canonical p-adic adelic metric. There's an ensuing interplay between height functions and height pairings. See §5 of [3]. There is no technical proof here. Instead, the goal of the talk is to summarize the results up to here and add them all together. Time permitting, the speaker can include a summary of the application to quadratic Chabauty from §7 of loc. cit.

2. Talk 3: Vologodsky functions and Besser's $\overline{\partial}$-operator

Let K be a finite extension of $\mathbb{Q}_p$, X smooth, geometrically connected over K. We'll work (mostly implicitly) with the K-Tannakian category uVIC(X) of unipotent vector bundles with integrable connection, and we'll make frequent use of the fact that any unipotent connection is analytically trivial in sufficiently small analytic polydisks. In the presence of a choice of value for the p-adic logarithm of p, which we fix once and for all, Vologodsky associates to $x, y \in X(K)$ a Tannakian path $v_{x,y} : x \to y$. Here are some of the properties:

(1) $v_{x,x} = \mathrm{Id}$.
(2) $v_{y,z} \cdot v_{x,y} = v_{x,z}$.
(3) If x, y are contained in a polydisk in which (E, ∇) is analytically trivial, then $v^E_{x,y} : E(x) \to E(y)$ agrees with parallel transport.

Definition 2.1. Let $\mathcal{F}$ be a vector bundle. An *abstract BV section of* $\mathcal{F}$ is a quadruple

$$(M, \nabla, s, v) = (\mathcal{F} \xleftarrow{s} M \xrightarrow{\nabla} \Omega^1_X \otimes M, \{v_x\}_x)$$

where M is a unipotent connection, $s : M \to \mathcal{F}$ is a morphism of $\mathcal{O}$-modules, and $v = \{v_x\}_x$ is a collection of vectors $v_x \in M(x)$ indexed by points with values in finite extensions of K such that

$$v_{x_1,x_2}(v_{x_1}) = v_{x_2}.$$

We also require an evident compatibility with field extensions. There's an evident notion of *morphism of abstract BV sections*. A *BV section* is a connected component of this category; we denote the BV section associated to (M, ∇, s, y) by $[M, \nabla, s, y]$. The resulting collection is denoted

$$\mathcal{F}_{\mathrm{Bes}}(X) = \mathcal{O}_{\mathrm{Bes}}(X, \mathcal{F}).$$

According to property (4) of Vologodsky paths, if U is a polydisk which trivializes (M, ∇), then there's a unique horizontal section v_U of (M, ∇) such that v_x is the value of v_U at x for each $x \in U$.

Given $f = [M, \nabla, s : M \to \mathcal{F}, v = \{v_x\}_x]$ and $x \in X(\overline{K})$, we define

$$f(x) := s(v_x) \in \mathcal{F}(x).$$

Straightforward constructions, including, for instance, maps

$$\mathcal{F}_{\mathrm{Bes}}(X) \times \mathcal{G}_{\mathrm{Bes}}(X) \to (\mathcal{F} \otimes \mathcal{G})_{\mathrm{Bes}}(X),$$

give the sets $\mathcal{O}_{\mathrm{Bes}}(X)$, $\mathcal{F}_{\mathrm{Bes}}(X)$ expected structures ($\mathcal{O}_X$-algebra, $\mathcal{O}_{\mathrm{Bes}}(X)$-module), and to homomorphisms to spaces of locally analytic sections. For instance, the homomorphism

$$\mathcal{O}(X) \to \mathcal{O}_{\mathrm{Bes}}(X)$$

is given by

$$f \mapsto (\mathcal{O}_X \xleftarrow{f} \mathcal{O}_X \xrightarrow{d} \Omega^1_X, 1)$$

and similarly for

$$\mathcal{F}(X) \to \mathcal{F}_{\mathrm{Bes}}(X). \tag{*}$$

We'll see soon that the map

$$\mathcal{F}_{\mathrm{Bes}} \to \mathcal{F}_{\mathrm{loc.an.}}(X)$$

to the space of locally analytic sections is injective, and it follows that (*) is also injective.

Example 2.2. The simplest examples are the trivial ones, that is, those Besser sections that are actually algebraic. These may be characterized as those Besser sections which admit a presentation $[\mathcal{F} \xleftarrow{s} E \xrightarrow{\nabla} \Omega^1 \otimes E, v]$ with (E, ∇) trivial. Indeed, under this assumption, the v_x assemble to a global horizontal section v_X. This gives rise to a morphism of abstract Besser sections as follows:

$$\begin{tikzcd} & \mathcal{O} \arrow[dl, "s(v)"'] \arrow[r] \arrow[d, "v"] & \Omega^1 \arrow[d, "v"] \\ \mathcal{F} & E \arrow[l, "s"] \arrow[r] & \Omega^1 \otimes E. \end{tikzcd}$$

Nontrivial examples arise from integration. In order to make sense of integration, we first need to define differentiation.

Suppose now that $\mathcal{F}$ too is endowed with a connection. Recall that if $s : M \to \mathcal{F} \otimes \Omega^i_X$ is a map of $\mathcal{O}$-modules, then

$$(\nabla(s) : M \to \mathcal{F} \otimes \Omega^{i+1}_X) \in \mathrm{Hom}(M, \mathcal{F} \otimes \Omega^{i+1}_X) \simeq \mathrm{Hom}(M, \mathcal{F}) \otimes \Omega^{i+1}_X$$

is given by

$$\nabla(s)(m) := \nabla(sm) - s(\nabla m)$$

(suitably interpreted).

Definition 2.3. Let $(\mathcal{F}, \nabla)$ be an integrable connection and $(M, \nabla, s : M \to \Omega^i_X \otimes \mathcal{F}, v)$ a BV-section of $\Omega^i_X \otimes \mathcal{F}$. We define

$$\nabla(M, \nabla, s : M \to \mathcal{F} \otimes \Omega^i_X, v) := [M, \nabla, \nabla s : M \to \mathcal{F} \otimes \Omega^{i+1}_X, v].$$

This gives rise to a complex

$$\mathcal{O}_{\mathrm{Bes}}(X, \mathcal{F}) \to \Omega^1_{\mathrm{Bes}}(X, \mathcal{F}) \to \Omega^2_{\mathrm{Bes}}(X, \mathcal{F}) \to \cdots$$

called the BV complex of $(\mathcal{F}, \nabla)$.

Lemma 2.4. If $f \in \mathcal{O}_{\mathrm{Bes}}(X)$ is a BV function and θ denotes passage to associated locally analytic sections, then

$$\theta(\nabla f) = d(\theta f).$$

Proof. Write $f = [M, \nabla, s : M \to \mathcal{O}, v]$. Fix U a sufficiently small analytic polydisk so that $M|_U$ is trivial. Then $\theta f|_U$ is the unique analytic function with values $(\theta f)(x) = s(v_x)$ and $d(\theta f)$ is given by applying the usual d.

Turning to the left hand side of the equation, we have that $\theta(\nabla f)|_U$ is the unique analytic 1-form with values

$$\theta(\nabla f)(x) = (\nabla s)(v_x).$$

The (algebraic, global) map of $\mathcal{O}_X$-modules

$$\nabla s : M \to \Omega^1_X$$

has the property that if v_U extends v_x to an analytic horizontal section over U, then

$$(\nabla s)(v_x) = ((\nabla s)(v_U))(x) = \big(d(sv_U) - s(\nabla v_U)\big)(x) = \big(d(sv_U)\big)(x)$$

which is the same as $d(\theta f)$ as described above. □

Proposition 2.5. For any polydisk $U \subset X$, the map

$$\mathcal{F}_{\mathrm{Bes}}(X) \to \mathcal{F}_{\mathrm{loc.an.}}(U)$$

is injective.

Sketch. Let $f = [M, \nabla, s : M \to \mathcal{F}, v = \{v_x\}_x]$ such that $s(v_x) = 0$ for all $x \in U$. After possibly shrinking U, we may assume (M, ∇) is trivial over U. We then have a unique analytic horizontal section $v_U \in M_{\mathrm{an}}(U)$ such that $s(v_U) = 0$ in $\mathcal{F}_{\mathrm{an}}(U)$. There's a maximal subconnection $(M_s, \nabla) \subset (M, \nabla)$ contained in the kernel of s:

$$M_s \subset \ker(s) \subset M.$$

Moreover, any (analytic) horizontal section of M contained in $\ker(s)$ is contained in M_s. In particular, $v_U \subset M_s(U)$. Applying the Tannakian Vologodsky paths to (M_s, ∇), we find that $v_x \in M_s(x)$ for all $x \in X$ (and in particular v_x is in the kernel of s even outside of U). We then get a hat (or span) of abstract Coleman functions as shown in the following diagram:

$$\begin{tikzcd}
 & M \arrow[r] \arrow[dl] & \Omega^1 \otimes M \\
\mathcal{F} & M_s \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \Omega^1 \otimes M_s \arrow[u] \arrow[d] \\
 & 0 \arrow[r] \arrow[ul] & \Omega^1 \otimes 0.
\end{tikzcd}$$

□

Theorem 2.6. The sequence

$$0 \to K \to \mathcal{O}_{\mathrm{Bes}}(X) \to \Omega^1_{\mathrm{Bes}}(X) \to \Omega^2_{\mathrm{Bes}}(X)$$

is exact.

Sketch. Suppose $f \in \mathcal{O}_{\mathrm{Bes}}(X)$ has zero differential. Then so does the associated locally analytic function, which is consequently equal to a constant c. Subtracting c and applying Proposition 2.5, we find that f may be represented by the abstract BV function $(\mathcal{O}, d, c : \mathcal{O} \to \mathcal{O}, 1)$, which is how the inclusion $K \to \mathcal{O}_{\mathrm{Bes}}(X)$ is defined.

Now suppose given $(\Omega^1 \xleftarrow{\omega} E \xrightarrow{\nabla} \Omega^1 \otimes E, \{v_x\}_x) \in \Omega^1_{\mathrm{Bes}}(X)$ such that

$$(d\omega)(v_x) = 0 \quad \text{for all } x \in X.$$

Consider the connection on $E \oplus \mathcal{O}_X$ defined by

$$\nabla(e, f) = (\nabla e, df - \omega e).$$

There's a *maximal integrable subconnection* $E' \subset E$ with the property that E contains all analytic horizontal sections. Fix U a polydisk which trivializes (E, ∇). Then we have

$$0 = (d\omega)(v_U) = d(\omega v_U).$$

So there's a locally analytic function g on U solving

$$\omega v_U = dg.$$

We set $w_U = (v_U, g)$ and we define w_x for $x \in X$ by transporting w_U along Vologodsky paths. Finally, we claim that

$$\nabla(\mathcal{O}_X \xleftarrow{\pi_2} E' \xrightarrow{\nabla} \Omega^1 \otimes E', \{w_x\}_x) \equiv (\Omega^1 \xleftarrow{\omega} E \xrightarrow{\nabla} \Omega^1 \otimes E, \{v_x\}_x)$$

as witnessed by the morphism of abstract BV 1-forms

$$\begin{tikzcd}
 & E' \arrow[r, "\nabla"] \arrow[dl, "\nabla\pi_2"'] \arrow[d, "\pi_1"] & \Omega^1 \otimes E' \arrow[d] \\
\Omega^1 & E \arrow[l, "\omega"'] \arrow[r, "\nabla"'] & \Omega^1 \otimes E.
\end{tikzcd}$$

□

Example 2.7. We may now give a nontrivial example of a BV-function. Let $X = \mathbb{G}_m$. We define log to be the primitive of $\frac{dz}{z}$ whose value at 1 is 0. (In particular, we retrieve the value $\log(p)$ which we fixed at the outset.) If we repeat the construction from the proof in this special case, we find that log may be represented by the abstract BV function

$$\log = [\mathcal{O} \xleftarrow{\pi_1} \mathcal{O}^2 \xrightarrow{\nabla} (\Omega^1)^2, (1, l)]$$

where

$$\nabla(e, f) = (de, df - e\frac{dz}{z})$$

and the family of local analytic functions l solves the differential equation

$$dl = \frac{dz}{z}.$$

Definition 2.8. We let $\mathcal{F}_1(X)$ be the set of BV-sections f of $\mathcal{F}$ such that f admits a representation $f = [\mathcal{F} \xleftarrow{s} E \xrightarrow{\nabla} \Omega^1 \otimes E, v]$ in which (E, ∇) is unipotent of level 2, i.e. fits in a short exact sequence

$$0 \to E_1 \to E \to E_2 \to 0$$

with E_1, E_2 trivial. We say that $f \in \mathcal{F}_{\mathrm{Bes}}(X)$ has *length 1* if $f \in \mathcal{F}_1(X)$.

Lemma 2.9. Assume X affine. Then any $f \in \mathcal{F}_1(X)$ may be decomposed as a sum

$$f = \sum (\int \omega_i) f_i$$

with $f_i \in \mathcal{F}(X)$ and $\omega_i \in \Omega^1(X)$.

Sketch. Fix $f = (\mathcal{F} \xleftarrow{s} E \xrightarrow{\nabla} \Omega^1 \otimes E, v) \in \mathcal{F}_1(X)$. We decompose into pieces in several ways and show that each piece is either of the form $g \in \mathcal{F}(X)$ or of the form $\int \eta$ for some $\eta \in \Omega^1(X)$.

Unipotent vector bundles on affine schemes are trivial. In particular $E \simeq \mathcal{O}^n$ and we may write

$$s = \sum_i g_i r_i$$

with $g_i \in \mathrm{Hom}(E, \mathcal{O})$ and $r_i \in \mathcal{F}(X)$. Let

$$G_i = (\mathcal{O} \xleftarrow{g_i} E \xrightarrow{\nabla} \Omega^1 \otimes E, v).$$

We've assumed that E sits in a short exact sequence of connections

$$\begin{tikzcd} 0 \arrow[r] & E_1 \arrow[r] & E \arrow[r, "\pi_2"] \arrow[l, bend right, dotted, "\pi_1"'] & E_2 \arrow[r] & 0 \end{tikzcd}$$

with E_1, E_2 trivial. After fixing a splitting π_1 on the level of underlying vector bundles as shown, we may write

$$g_i = s_1\pi_1 + s_2\pi_2$$

with $s_i \in \mathrm{Hom}(E_i, \mathcal{O})$. Since π_2 is horizontal, we have

$$[\mathcal{O} \xleftarrow{s_2} E_2 \xleftarrow{\pi_2} E \xrightarrow{\nabla} \Omega^1 \otimes E, v] \equiv [\mathcal{O} \xleftarrow{s_2} E_2 \xrightarrow{\nabla} \Omega^1 \otimes E_2, \pi_2(v)]$$

which belongs to $\mathcal{O}(X)$. On the other hand, after possibly multiplying by an algebraic function, we may assume that s_1 is horizontal. In this case, $\nabla(s_1\pi_1)$ vanishes on E_1 and therefore equals $\omega_2\pi_2$ for some

$$\omega_2 : E_2 \to \Omega^1.$$

Consequently,

$$d[\mathcal{O} \xleftarrow{s_1} E_1 \xleftarrow{\pi_1} E \xrightarrow{\nabla} \Omega^1 \otimes E, v] = [\Omega^1 \xleftarrow{\omega_2} E_2 \xrightarrow{\nabla} \Omega^1 \otimes E_2, \pi_2(v)] = \eta$$

and the latter belongs to $\Omega^1(X)$. Thus,

$$[\mathcal{O} \xleftarrow{s_1} E_1 \xleftarrow{\pi_1} E \xrightarrow{\nabla} \Omega^1 \otimes E, v] = \int \eta$$

is an antiderivative of the holomorphic 1-form η. □

Definition 2.10. Let $f = [\mathcal{F} \xleftarrow{s} E \xrightarrow{\nabla} \Omega^1 \otimes E, v] \in \mathcal{F}_1(X)$ be a BV-section of length 1, and let

$$0 \to E_1 \to E \to E_2 \to 0$$

be a short exact sequence of connections. The image of the v assemble to a global horizontal section v^2 of E_2; pullback along v^2 gives us an extension

$$\mathcal{E}' = (0 \to E_1 \to E' \to \mathcal{O} \to 0).$$

We transport the restriction s^1 of s to E_1 along the isomorphism

$$\mathrm{Hom}(E_1, \mathcal{F}) \simeq \mathrm{Hom}(\mathcal{O}, \mathcal{F}) \otimes \mathrm{Hom}_\nabla(E_1, \mathcal{O})$$

and then use the map

$$\mathrm{Hom}_\nabla(E_1, \mathcal{O}) \to \mathrm{Ext}^1_\nabla(\mathcal{O}, \mathcal{O})$$

induced by the extension class of $\mathcal{E}'$ to obtain an element

$$\overline{\partial} f \in \mathrm{Ext}^1_\nabla(\mathcal{O}, \mathcal{O}) \otimes \mathrm{Hom}(\mathcal{O}, \mathcal{F}) = H^1_{\mathrm{dR}}(X) \otimes \mathcal{F}(X).$$

Besser uses a notion of *minimal representative* to show that this is well defined and gives rise to a homomorphism

$$\overline{\partial} : \mathcal{F}_1(X) \to H^1_{\mathrm{dR}}(X) \otimes \mathcal{F}(X).$$

Actually, Besser recommends replacing $\overline{\partial}$ by $-\overline{\partial}$.

Example 2.11. Suppose X affine, $F \in \mathcal{O}_1(X)$, $f \in \mathcal{F}(X)$. Consider the abstract BV-section $(\mathcal{O}^2, v, s)$ given by

$$\mathcal{F} \xleftarrow{f \cdot \pi_2} \mathcal{O}^2 \xrightarrow{\begin{pmatrix} 0 & -dF \\ 0 & 0 \end{pmatrix}} (\Omega^1)^2, \quad v = (1, F).$$

Thus

$$[\mathcal{O}^2, v, s] \equiv Ff \in \mathcal{F}_1(X).$$

Applying our construction of $\overline{\partial}$ we find that

$$\overline{\partial}(Ff) = [dF] \otimes f.$$

Theorem 2.12. There is a short exact sequence

$$0 \to \mathcal{F}(X) \to \mathcal{F}_1(X) \xrightarrow{\overline{\partial}} H^1_{\mathrm{dR}}(X) \otimes \mathcal{F}(X).$$

If X is affine, then $\overline{\partial}$ is surjective.

Sketch. Besser uses a notion of "minimal representatives" to show that $\mathcal{F}_{\mathrm{Bes}}$, $\mathcal{F}_1$ are Zariski sheaves. The statement can then be reduced to the case that X is affine. The surjectivity then follows from the construction of example 2.12.

Suppose $f \in \ker \overline{\partial}$. Using Lemma 2.9, write

$$f = \sum f_i \int \eta_i$$

with $f_i \in \mathcal{F}(X)$ and $\eta_i \in \Omega^1(X)$. By example 2.11,

$$0 = \overline{\partial} f = \sum f_i \otimes [\eta_i].$$

We can arrange things so that the f_i are linearly independent over K, so that $[\eta_i] = 0$ and hence $\int \eta_i \in \mathcal{O}(X)$. Thus, $f \in \mathcal{F}(X)$. □

3. Talk 7: Global p-adic heights and quadratic Chabauty

3.1. **Valuations and log functions.** We'll work over $K = \mathbb{Q}$; this will allow us to skip a few technicalities while nevertheless preserving the main ideas. We let X be a smooth, projective, geometrically integral $\mathbb{Q}$-scheme. Let L be a line bundle. A *valuation on* $L_{\overline{\mathbb{Q}}_p}$ is a function

$$v_L : L^\times(\overline{\mathbb{Q}}_p) \to \overline{\mathbb{Q}}_p$$

such that

$$v_L(\lambda u) = \mathrm{ord}_p(\lambda) + v_L(u)$$

for all $u \in L^\times(\overline{\mathbb{Q}}_p)$ and all $\lambda \in \overline{\mathbb{Q}}_p^*$.

When X is an abelian variety, by theorem 9.5.7 of [5], after choosing a rigidification $r \in L^\times(K)$, there's a canonical valuation on $L_{\overline{\mathbb{Q}}_p}$ at every finite place $\mathfrak{p}$ of K. These satisfy a list of desiderata which I do not reproduce here. Especially important is that v_L is $\mathbb{Q}$-valued and $v_L(r) = 0$. See Proposition 2.9 of [3].

We recall that a *log function on* $L_{K_\mathfrak{p}}$ is a Besser-Vologodsky function $\log_L \in \mathcal{O}_{\mathrm{Bes}}(L^\times_{K_\mathfrak{p}})$ such that

$$d\log_L \in \Omega_1(L^\times_{K_\mathfrak{p}}), \quad \text{and} \quad \log_L(\lambda u) = \log(\lambda) + \log_L(u)$$

for any $u \in L^\times(\overline{K}_\mathfrak{p})$ and any $\lambda \in \overline{K}_\mathfrak{p}$. We say that $\alpha \in \Omega^1(X_{K_\mathfrak{p}}) \otimes H^1_{\mathrm{dR}}(X_{K_\mathfrak{p}})$ is a *curvature form for* $L_{K_\mathfrak{p}}$ if

$$\cup\alpha = c_1(L_{K_\mathfrak{p}}).$$

We'll say that a curvature form α for $L_{K_{\mathfrak{p}}}$ and a log function $\log_L$ are *associated* if

$$\pi^*\alpha = \bar{\partial} d \log_L$$

where π denotes the map $L \to X$. A log function uniquely determines an associated curvature form. In the opposite direction, every curvature form has an associated log function, and determines the associated log function up to the integral of a holomorphic form.

Valuations and log functions on line bundles induce valuations and log functions on tensor products.

3.2. **The height function associated to a p-adic adelic metric.** A *p-adic adelic metric on L* consists of the following data. For every place $\mathfrak{p}|p$, a log function $\log_{L,\mathfrak{p}}$ on $L_{K_{\mathfrak{p}}}$. For every finite place $\mathfrak{q} \nmid p$, a $\mathbb{Q}$-valued valuation $v_{L,\mathfrak{q}}$ on $L_{K_{\mathfrak{q}}}$. These are required to satisfy the following compatibility condition: There exist integral models $\mathcal{X}$, $\mathcal{L}$ over $\mathcal{O}_K$ such that at all but finitely many places $\mathfrak{q} \nmid p$, the valuation $v_{L,\mathfrak{q}}$ should be the associated "model valuation" given by naive intersection with the 0-section.

We make use now of our assumption that actually $K = \mathbb{Q}$. There exists a continuous homomorphism

$$\chi = \sum : \mathbb{A}_{\mathbb{Q}}^*/\mathbb{Q}^* \to \mathbb{Q}_p$$

with the following properties. We denote by χ_q the composition of χ with the evident map

$$\mathbb{Q}_q^* \to \mathbb{A}_{\mathbb{Q}}^*/\mathbb{Q}^*.$$

We require that for $q \neq p$, $\chi_q(\mathbb{Z}_q^*) = 0$ and that

$$\chi_p = \log_p : \mathbb{Q}_p^* \to \mathbb{Q}_p$$

be a branch of the p-adic logarithm. We can further arrange for the branch of the logarithm to be given by $\log_p(p) = 0$.

Since we'll often consider base change to $K_{\mathfrak{q}}$ and rarely (or never) consider base-change to its residue field, we'll allow ourselves to write '$X_{\mathfrak{q}}$' as an abbreviation for $X_{K_{\mathfrak{q}}}$.

The *p-adic height function*

$$X(\mathbb{Q}) \to \mathbb{Q}_p$$

is given by

$$h(x) = \log_{L,p}(u) + \sum_{q \neq p} v_{L,q}(u)\chi_q(q).$$

for any $u \in L(x) \setminus \{0\}$.

3.3. **Log functions on abelian varieties.** We now set $X = A$ an abelian variety over K. Everything will take place over the local field $K_{\mathfrak{p}}$ at a p-adic place $\mathfrak{p}$. A log function on a rigidified line bundle (L, r) is *normalized* if $\log_L(r) = 0$. For L symmetric, a log function is *good* if the isomorphism

("19") $$s^* L \otimes d^* L \simeq (\pi_1^* L)^{\otimes 2} \otimes (\pi_2^* L)^{\otimes 2}$$

is an *isometry*, i.e. compatible with induced log functions. For L antisymmetric, a log function is *good* if the isomorphism

("20") $$s^* L \simeq \pi_1^* L \otimes \pi_2^* L$$

is an isometry.

We say $\alpha \in \Omega^1(A \times \hat{A}) \otimes H^1_{\mathrm{dR}}(A \times \hat{A})$ is *purely mixed* if it's contained in the summand

$$\Omega^1(A) \otimes H^1_{\mathrm{dR}}(\hat{A}) \ \oplus\ \Omega^1(\hat{A}) \otimes H^1_{\mathrm{dR}}(A) \ \subset\ \Omega^1(A \times \hat{A}) \otimes H^1_{\mathrm{dR}}(A \times \hat{A}).$$

Let $P \to A \times \hat{A}$ be the Poincaré bundle. We fix arbitrarily a rigidification r_P of P over the origin $(0, 0)$. This choice induces a trivialization

$$P_{\{0_A\} \times \hat{A}} \simeq \mathbb{A}^1_{\hat{A}}.$$

In turn, for any $\hat{a} \in \hat{A}$, the trivialization induces a trivialization of the fiber of $P_{A \times \{\hat{a}\}}$ above $0 \in A$, and hence an associated rigidification $r_{\hat{a}}$ of $P_{A \times \{\hat{a}\}}$.

There is a unique log function $\log_P$ on P with curvature form α which restricts to the trivial log function on $A \times \{0\}$ and on $\{0\} \times \hat{A}$. Any rigidified antisymmetric line bundle is uniquely isomorphic to a unique rigidified fiber

$$(L, r) \simeq (P_{A \times \{\hat{a}\}}, r_{\hat{a}}).$$

We define the *canonical log function on L* to be the one induced form $\log_P$.

If $L \to A$ is a line bundle, we have the associated polarization

$$\phi_L : A \to \hat{A}$$

given on points by $\phi_L(a) = t_a^* L \otimes L^{-1}$. If L is symmetric then there's an isomorphism

$$(\mathrm{id} \times \phi_L)^* P \simeq L^{\otimes 2}.$$

In this case, there's a unique good log function with curvature $\frac{1}{2}(\mathrm{id} \times \phi_L)^* \alpha$. This defines the *canonical log function* for L symmetric.

For an arbitrary line bundle, setting

$$L^+ = L \otimes [-1]^* L, \quad L^- = L \otimes ([-1]^* L)^{-1}$$

we have a decomposition

("30") $$L^{\otimes 2} \simeq L^+ \otimes L^-$$

into symmetric and antisymmetric parts. We define the *canonical log function* $\log_{L^{\otimes 2}}$ by

$$\log_{L^{\otimes 2}}(s \otimes t) = \log_{L^+}(s) + \log_{L^-}(t)$$

and on $\log_L$ by

$$\log_L(s) = \frac{1}{2}\log_{L^{\otimes 2}}(s^{\otimes 2}).$$

3.4. **Heights on abelian varieties.** Fix $A/\mathbb{Q}$ an abelian variety and L a line bundle.

Proposition 3.5. The height function associated to the canonical adelic metric on L is quadratic.

Proof. We address first the case that L is symmetric. In this case, isomorphism 19 preserves adelic metrics, and we therefore have an equality of associated height functions

$$h_{s^*L\otimes d^*L} = h_{\pi_1^*L^{\otimes 2}\otimes \pi_2^*L^{\otimes 2}}.$$

Fix $(x, y) \in A \times A(K)$. Then

$$\begin{aligned}(s^*L \otimes d^*L)(x, y) &= (s^*L)(x, y) \otimes (d^*L)(x, y)\\ &= L(s(x, y)) \otimes L(d(x, y))\\ &= L(x + y) \otimes L(x - y).\end{aligned}$$

We may therefore choose a vector of the form $v \otimes u$ with $v \in L(x + y)$ and $u \in L(x - y)$ and compute

$$\begin{aligned}h_{s^*L\otimes d^*L}(x, y) &= \log_{s^*L\otimes d^*L,p}(v \otimes u) + \sum_{q\neq p} v_{s^*L\otimes d^*L,q}(v \otimes u)\chi_q(q)\\ &= \log_{L,p}(v) + \log_{L,p}(u) + \sum_{q\neq p}\left[v_{L,q}(v) + v_{L,q}(u)\right]\chi_q(q)\\ &= h_L(x + y) + h_L(x - y).\end{aligned}$$

A similar calculation shows that

$$h_{\pi_1^*L^{\otimes 2}\otimes\pi_2^*L^{\otimes 2}}(x, y) = 2h(x) + 2h(y).$$

Together, we have

$$h_L(x + y) + h_L(x - y) = 2h_L(x) + 2h_L(y).$$

We also have $h_L(0) = 0$. Together, via elementary manipulations, these imply that h_L is a homogeneous quadratic form in this case. See e.g. p. 98 of Lang's *Fundamentals*.

Similar logic using equation 20 in place of 19 shows that the height function associated to an antisymmetric line bundle is linear. For a general line bundle we have that equation 30 is an isometry, which shows that the associated height function is a linear combination of linear and quadratic parts. ◻

3.6. **Quadratic Chabauty.** We now assume $X/\mathbb{Q}$ is a curve (still smooth and projective) of genus $g \geq 2$. We assume $X(\mathbb{Q})$ contains a point b. We fix an arbitrary prime p not assumed to be of good reduction (but we do impose a condition on p below). We fix a purely mixed curvature form on P_p, the base change to $\mathbb{Q}_p$ of the Poincaré bundle of the Jacobian J. We denote the embedding

$$X \to J$$

associated to b by ι. We restrict attention to the case $\operatorname{rk} J(\mathbb{Q}) = g$.

We assume there exists a line bundle L on J which maps to a nonzero element of the Néron-Severi group such that $\iota^* L \simeq \mathbb{A}^1_X$, and we fix such an isomorphism, which we think of also as a map of total spaces

$$\begin{tikzcd} \mathbb{A}^1_X \arrow[r, "\tilde{\iota}"] \arrow[d] & L \arrow[d] \\ X \arrow[r, "\iota"'] \arrow[u, bend left, "1"] & J. \end{tikzcd}$$

We denote the canonical section by '1', as shown, and we denote the image of $x \in X$ by 1_x. The section 1 and the trivialization $\tilde{\iota}$ together provide us with an associated rigidification $r_b \in L(\tilde{\iota}(1_b))$.

For $q \neq p$, we define

$$\lambda_{L,q} : X(\mathbb{Q}_q) \to \mathbb{Q}$$

by

$$\lambda_{L,q}(x) = v_{L,q}(\tilde{\iota} 1_x).$$

Then λ_q takes on only finitely many values, and is identically 0 if X has potentially good reduction at q (properties of canonical valuations, see e.g. [5]). Hence, the sets

$$\Lambda_{L,q} = \{\lambda_{L,q}(x) \mid x \in X(\mathbb{Q}_q)\} \subset \mathbb{Q}$$

and

$$\Lambda_L = \{\sum_{q \neq p} l_q \chi_q(q) \mid l_q \in \Lambda_{L,q}\} \subset \mathbb{Q}_p$$

are finite.

By compatibility of coherent cohomology with flat base change, we have

$$H^0(J, \Omega^1) \otimes \mathbb{Q}_p \simeq H^0(J_{\mathbb{Q}_p}, \Omega^1).$$

The map

$$J(\mathbb{Q}) \otimes \mathbb{Q}_p \to J(\mathbb{Q}_p) \otimes \mathbb{Q}_p$$

is, by our assumptions, a map of vector spaces of same dimension $r = g$. We assume it's an isomorphism. (If it isn't, then linear Chabauty applies.)

The pairing

$$H^0(J_{\mathbb{Q}_p}, \Omega^1) \times J(\mathbb{Q}_p) \to \mathbb{Q}_p$$
$$(\omega, x) \mapsto \int_0^x \omega$$

induces an isomorphism

$$H^0(J_{\mathbb{Q}_p}, \Omega^1) \simeq (J(\mathbb{Q}_p) \otimes \mathbb{Q}_p)^\vee.$$

(Facts concerning the p-adic logarithm map of an abelian variety well known to some. This implicitly includes the statement that the single integrals as defined in previous lectures predate Vologodsky by many years.)

Combining the above, we have an isomorphism

$$H^0(J, \Omega^1) \otimes \mathbb{Q}_p \simeq (J(\mathbb{Q}) \otimes \mathbb{Q}_p)^\vee.$$

Via this isomorphism, a basis $\omega_1, \ldots, \omega_g$ of the $\mathbb{Q}$-*vector space* $H^0(J, \Omega^1)$ gives rise to a basis $f_1, \ldots, f_g$ of $(J(\mathbb{Q}) \otimes \mathbb{Q}_p)^\vee$ where each f_i may be identified with the Vologodsky function

$$f_i(x) = \int_0^x \omega_i$$

on $J(\mathbb{Q}_p)$.

Let $(\hat{h}_L)_{\mathbb{Q}_p}$ be the unique quadratic extension of the canonical height to $J(\mathbb{Q}) \otimes \mathbb{Q}_p$. Then there are uniquely determined p-adic numbers $a_{ij}, b_k \in \mathbb{Q}_p$ such that

$$(\hat{h}_L)_{\mathbb{Q}_p} = \sum a_{ij} f_i f_j + \sum b_k f_k.$$

The local terms of the canonical height $\hat{h}_L(x)$ for $x \in J(\mathbb{Q})$ depend on the auxiliary choice of a vector $u \in L(x)$. If $x \in X(\mathbb{Q}) \subset J(\mathbb{Q})$, we may choose

$$u = 1_x.$$

As a matter of notation, let's identify X, $\mathbb{A}^1_X$ with their images under $\iota, \tilde{\iota}$. We then find that for $x \in X(\mathbb{Q})$,

$$\hat{h}_L(x) = \log_{L,p}(1_x) + \sum_{q \neq p} \lambda_{L,q}(x) \chi_q(q).$$

Finally, we let

$$F : X(\mathbb{Q}_p) \to \mathbb{Q}_p$$

be the Besser-Vologodsky function given by

$$F(x) = \sum a_{ij} f_i(x) f_j(x) + \sum b_k f_k(x) - \log_{L,p}(1_x)$$

and we define

$$X(\mathbb{Q}_p)^L_{\text{Quadratic}} = \{x \in X(\mathbb{Q}_p) \mid F(x) \in \Lambda_L\}.$$

By construction, we have

$$X(\mathbb{Q}) \subset X(\mathbb{Q}_p)^L_{\text{Quadratic}}.$$

4. Exercises

(1) How much does $X(\mathbb{Q}_p)^L_{\text{Quadratic}}$ depend on L? Why does L need to have nontrivial Néron-Severi class? Note that there's no choice of L in Chabauty-Kim theory.

(2) Speaking of Chabauty-Kim theory, find a finite type quotient U of the unipotent fundamental group such that (at least for p of good reduction)

$$X(\mathbb{Q}_p)_U = X(\mathbb{Q}_p)^L_{\text{Quadratic}}.$$

(3) This is just as much a question about real heights as it is about p-adic heights. Intuitively, the local terms of the height function associated to an adelic metric on a line bundle are given roughly by a naive intersection multiplicity of the given vector (regarded as a 1-cycle in a $\mathbb{Z}_p$-model) against the zero-section. This is precise for the valuations at primes of good reduction, imprecise but also not too far from the truth for valuations at primes of bad reduction (true up to a constant on each component of the special fiber, at least for a good choice of model), and merely a naive and possibly faulty intuition at p.

Meanwhile, the global canonical height $\hat{h}_L(x)$ of a point $x \in A(\mathbb{Q})$ for A an abelian variety and L a line bundle, is supposed to be given by an intersection multiplicity

$$\langle x, t_x^* D - D\rangle \tag{*}$$

of x regarded as a 1-cycle against a divisor associated to $\phi_L(x)$ (both living in the Néron model) at least up to some linear factor. The appearance of x on both sides accounts for the quadraticity.

Make these statements precise, if they can be made precise. Explain intuitively why the sum of the local intersection multiplicities gives rise to (*).

(4) How do you think intuitively about an intersection multiplicity that's necessarily and irrevocably $\mathbb{R}$-valued or $\mathbb{Q}_p$-valued?

(5) How can the same space be compactified near different points (i.e. near different primes)?

(6) Related to the previous question. Intuition for the real component of real heights is said to come from potential theory in physics. Explain this. What's the p-adic analog of potential theory?

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